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# An EP theorem for dual linear complementarity problems

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## Abstract

The linear complementarity problem (*LCP*) belongs to the class of  $\text{NP}$ -complete problems. Therefore we can not expect a polynomial time solution method for *LCP*s without requiring some special property of the matrix of the problem. We show that the dual *LCP* can be solved in polynomial time if the matrix is row sufficient, moreover in this case all feasible solutions are complementary. Furthermore we present an existentially polytime (EP) theorem for the dual *LCP* with arbitrary matrix.

*Keywords:* Linear complementarity problem, row sufficient matrix,  $\mathcal{P}_*$ -matrix, EP theorem

## 1 Introduction

Consider the *linear complementarity problem (LCP)*: find vectors  $\mathbf{x}, \mathbf{s} \in \mathbb{R}^n$ , that satisfy the constraints

$$-M\mathbf{x} + \mathbf{s} = \mathbf{q}, \quad \mathbf{x}\mathbf{s} = \mathbf{0}, \quad \mathbf{x}, \mathbf{s} \geq \mathbf{0}, \quad (1)$$

where  $M \in \mathbb{R}^{n \times n}$  and  $\mathbf{q} \in \mathbb{R}^n$ , and the notation  $\mathbf{x}\mathbf{s}$  is used for the coordinatewise (Hadamard) product of the vectors  $\mathbf{x}$  and  $\mathbf{s}$ .

Problem (*LCP*) belongs to the class of  $\text{NP}$ -complete problems, since the feasibility problem of linear equations with binary variables can be described as an *LCP* [9]. Therefore we can not expect an efficient (polynomial time) solution method for *LCP*s without requiring some special property of the matrix  $M$ . The matrix classes that are important for our goals are discussed in Section 2, along with the *LCP* duality theorem and an EP form of the duality theorem.

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Consider the *dual linear complementarity problem (DLCP)* [5, 6]: find vectors  $\mathbf{u}, \mathbf{z} \in \mathbb{R}^n$ , that satisfy the constraints

$$\mathbf{u} + M^T \mathbf{z} = \mathbf{0}, \quad \mathbf{q}^T \mathbf{z} = -1, \quad \mathbf{u} \mathbf{z} = \mathbf{0}, \quad \mathbf{u}, \mathbf{z} \geq \mathbf{0}. \quad (2)$$

We show that the dual *LCP* can be solved in polynomial time if the matrix is row sufficient, as for this case all feasible solutions are complementary (see Lemma 6). This result yields an improvement compared to earlier known polynomial time complexity results, namely an *LCP* is solvable in polynomial time for  $\mathcal{P}_*(\kappa)$ -matrices with known  $\kappa \geq 0$ . Due to the special structure of (*DLCP*), the polynomial time complexity of interior point methods depends on the row sufficient property of the coefficient matrix  $M$ . Furthermore, we present an EP theorem for the dual *LCP* with arbitrary matrix  $M$ , and apply the results for homogeneous *LCP*'s.

Throughout the paper the following notations are used. Scalars and indices are denoted by lowercase Latin letters, vectors by lowercase boldface Latin letters, matrices by capital Latin letters, and finally sets by capital calligraphic letters. Further,  $\mathbb{R}_{\oplus}^n$  ( $\mathbb{R}_+^n$ ) denotes the nonnegative (positive) orthant of  $\mathbb{R}^n$ , and  $X$  denotes the diagonal matrix whose diagonal elements are the coordinates of vector  $\mathbf{x}$ , i.e.,  $X = \text{diag}(\mathbf{x})$  and  $I$  denotes the identity matrix of appropriate dimension. The vector  $\mathbf{x} \mathbf{s} = X \mathbf{s}$  is the componentwise product (Hadamard product) of the vectors  $\mathbf{x}$  and  $\mathbf{s}$ , and for  $\alpha \in \mathbb{R}$  the vector  $\mathbf{x}^\alpha$  denotes the vector whose  $i$ th component is  $x_i^\alpha$ . We denote the vector of ones by  $\mathbf{e}$ . Furthermore

$$\mathcal{F}_P = \{(\mathbf{x}, \mathbf{s}) \geq \mathbf{0} : -M\mathbf{x} + \mathbf{s} = \mathbf{q}\}$$

is the set of the feasible solutions of problem (*LCP*) and

$$\mathcal{F}_D = \{(\mathbf{u}, \mathbf{z}) \geq \mathbf{0} : \mathbf{u} + M^T \mathbf{z} = \mathbf{0}, \mathbf{q}^T \mathbf{z} = -1\}$$

is the set of the feasible solutions of problem (*DLCP*).

The rest of the paper is organized as follows. The following section reviews the necessary definitions and basic properties of the matrix classes used in this paper. In Section 3 we present our main results about polynomial time solvability of dual *LCP*'s.

## 2 Matrix classes and *LCP*'s

The class of  $P_*(\kappa)$ -matrices, that can be considered as a generalization of the class of positive semidefinite matrices, were introduced by Kojima et al. [9].

**Definition 1** *Let  $\kappa \geq 0$  be a nonnegative number. A matrix  $M \in \mathbb{R}^{n \times n}$  is a  $\mathcal{P}_*(\kappa)$ -matrix if*

$$(1 + 4\kappa) \sum_{i \in \mathcal{I}_+(\mathbf{x})} x_i(Mx)_i + \sum_{i \in \mathcal{I}_-(\mathbf{x})} x_i(Mx)_i \geq 0, \quad \text{for all } \mathbf{x} \in \mathbb{R}^n,$$

where  $\mathcal{I}_+(\mathbf{x}) = \{1 \leq i \leq n : x_i(Mx)_i > 0\}$  and  $\mathcal{I}_-(\mathbf{x}) = \{1 \leq i \leq n : x_i(Mx)_i < 0\}$ .

The nonnegative number  $\kappa$  denotes the weight that need to be used at the positive terms so that the weighted 'scalar product' is nonnegative for each vector  $\mathbf{x} \in \mathbb{R}^n$ . Therefore,  $\mathcal{P}_*(0)$  is the class of positive semidefinite matrices (if we set aside the symmetry of the matrix  $M$ ).

**Definition 2** A matrix  $M \in \mathbb{R}^{n \times n}$  is a  $\mathcal{P}_*$ -matrix if it is a  $\mathcal{P}_*(\kappa)$ -matrix for some  $\kappa \geq 0$ , i.e.

$$\mathcal{P}_* = \bigcup_{\kappa \geq 0} \mathcal{P}_*(\kappa).$$

The class of sufficient matrices was introduced by Cottle, Pang and Venkateswaran [2].

**Definition 3** A matrix  $M \in \mathbb{R}^{n \times n}$  is a column sufficient matrix if for all  $\mathbf{x} \in \mathbb{R}^n$

$$X(M\mathbf{x}) \leq 0 \text{ implies } X(M\mathbf{x}) = 0,$$

and it is row sufficient if  $M^T$  is column sufficient. The matrix  $M$  is sufficient if it is both row and column sufficient.

Kojima et al. [9] proved that any  $\mathcal{P}_*$  matrix is column sufficient and Guu and Cottle [7] proved that it is row sufficient too. Therefore, each  $\mathcal{P}_*$  matrix is sufficient. Väliaho proved the other direction of inclusion [11], thus the class of  $\mathcal{P}_*$ -matrices coincides with the class of sufficient matrices.

Fukuda and Terlaky [6] proved a fundamental theorem for quadratic programming in oriented matroids. As they stated in their paper, the *LCP* duality theorem follows from that theorem for sufficient matrix *LCPs*.

**Theorem 4** Let a sufficient matrix  $M \in \mathbb{Q}^{n \times n}$  and a vector  $\mathbf{q} \in 0\mathbb{Q}^n$  be given. Then exactly one of the following statements hold:

- (1) problem (*LCP*) has a solution  $(\mathbf{x}, \mathbf{s})$  whose encoding size is polynomially bounded.
- (2) problem (*DLCP*) has a solution  $(\mathbf{u}, \mathbf{v})$  whose encoding size is polynomially bounded.

A direct and constructive proof of the *LCP* duality theorem can be found in [4].

The concept of EP (existentially polynomial-time) theorems was introduced by Cameron and Edmonds [1]. It is a theorem of the form:

$$[\forall x : F_1(x), F_2(x), \dots, F_k(x)],$$

where  $F_i(x)$  is a predicate formula which has the form

$$F_i(x) = [\exists y_i \text{ such that } \|y_i\| \leq \|x\|^{n_i} \text{ and } f_i(x, y_i)].$$

Here  $n_i \in \mathbb{Z}^+$ ,  $\|z\|$  denotes the encoding length of  $z$  and  $f_i(x, y_i)$  is a predicate formula for which there is a polynomial size certificate.

The *LCP* duality theorem in EP form was given by Fukuda, Namiki and Tamura [5]:

**Theorem 5** Let a matrix  $M \in \mathbb{Q}^{n \times n}$  and a vector  $\mathbf{q} \in \mathbb{Q}^n$  be given. Then at least one of the following statements hold:

- (1) problem (LCP) has a complementary feasible solution  $(\mathbf{x}, \mathbf{s})$ , whose encoding size is polynomially bounded.
- (2) problem (DLCP) has a complementary feasible solution  $(\mathbf{u}, \mathbf{z})$ , whose encoding size is polynomially bounded.
- (3) matrix  $M$  is not sufficient and there is a certificate whose encoding size is polynomially bounded.

### 3 Main results

In this section we show that if the matrix is row sufficient then all feasible solutions of (DLCP) are not only nonnegative, but they are complementary as well. Based on this result we get an EP theorem for problem (DLCP).

**Lemma 6** *Let matrix  $M$  be row sufficient. If  $(\mathbf{u}, \mathbf{z}) \in \mathcal{F}_D$ , then  $(\mathbf{u}, \mathbf{z})$  is a solution of problem (DLCP).*

**Proof:** The vector  $(\mathbf{u}, \mathbf{z})$  is a feasible solution of problem (DLCP), therefore  $\mathbf{u}, \mathbf{z} \geq \mathbf{0}$  and  $\mathbf{u} = -M^T \mathbf{z}$ , so the complementarity gap is nonnegative

$$\mathbf{0} \leq \mathbf{u} \mathbf{z} = -\mathbf{z} M^T \mathbf{z} = -Z M^T \mathbf{z}.$$

From here, and by Definition 3, if matrix  $M$  is a row sufficient matrix, then  $Z M^T \mathbf{z} = \mathbf{0}$ , thus  $\mathbf{u} \mathbf{z} = \mathbf{0}$ . ■

**Corollary 7** *Let matrix  $M$  be row sufficient. Then problem (DLCP) can be solved in polynomial time*

**Proof:** By Lemma 6, if  $M$  is row sufficient, one need only to solve the feasibility problem of (DLCP), that is one need to solve only a linear feasibility problem what can be done in polynomial time e.g., by interior point methods [10]. ■

We have to note that there is no known polynomial time algorithm for checking whether a matrix is row sufficient or not. The following theorem presents what can be proved about an LCP problem with arbitrary matrix using a polynomial time algorithm.

**Theorem 8** *Let matrix  $M \in \mathbb{Q}^{n \times n}$  and vector  $\mathbf{q} \in \mathbb{Q}^n$  be given. Then it can be shown in polynomial time that at least one of the following statements hold:*

- (1) problem (DLCP) has a feasible complementary solution  $(\mathbf{u}, \mathbf{z})$ , whose encoding size is polynomially bounded.
- (2) problem (LCP) has a feasible solution, whose encoding size is polynomially bounded.
- (3) matrix  $M$  is not row sufficient and there is a certificate whose encoding size is polynomially bounded.

**Proof:** Apply a polynomial time algorithm to solve the feasibility problem of  $(DLCP)$ , i.e., to find a point in the set  $\mathcal{F}_D$ . This is a linear feasibility problem, thus it can be done in polynomial time with e.g., interior point methods using the self-dual embedding technique (see [10]). If  $\mathcal{F}_D = \emptyset$ , then by the Farkas Lemma  $\mathcal{F}_P \neq \emptyset$ , and a primal feasible point can be read out from the solution of the embedding problem, thus we get the second case. Otherwise, we get a point  $(\mathbf{u}, \mathbf{z}) \in \mathcal{F}_D$ . If the complementarity condition,  $\mathbf{u}\mathbf{z} = \mathbf{0}$  holds too, then the point  $(\mathbf{u}, \mathbf{z})$  is a solution of problem  $(DLCP)$ , so we get the first case. Finally if the feasible solution  $(\mathbf{u}, \mathbf{z})$  is not complementary, then according to Lemma 6 vector  $\mathbf{z}$  provides a certificate that matrix  $M$  is not a row sufficient matrix. As the encoding size of the solution of the self-dual embedding problem, after a proper rounding procedure, is polynomially bounded, the third option holds in this case. ■

Observe that Theorem 8 is in EP form. Both Theorems 5 and 8 deal with problem  $(LCP)$ , but Theorem 5 approaches the problem from the primal, while Theorem 8 from the dual side. The advantages of Theorem 8 is to determine certificates in polynomial time. The proof of Theorem 5 is constructive too, it is based on the criss-cross algorithm (for details see [4, 5]). The  $LCP$  duality theorem gives in the first two cases not only a feasible, but also complementary solutions. We deal with the second case of Theorem 8 in the paper [8], where some modified interior point methods are presented that either solve problem  $(LCP)$  with the given arbitrary matrix, or provide a polynomial size certificate in polynomial time, that the matrix of the problem is not sufficient.

Finally, let us note that using the technique presented in this paper, the following result can be obtained for a special class of  $LCP$ 's as well.

**Theorem 9** *Let matrix  $M$  be column sufficient and  $\mathbf{q} = \mathbf{0}$ . Then every feasible solution of  $(LCP)$  is complementarity. Furthermore, if there is a nontrivial solution, it can be found in polynomial time.*

**Proof:** Analogous to the proof of Lemma 6, we can show the first statement, namely that the set of feasible solutions coincides with the set of solutions. If there is a nontrivial solution, then there is a nontrivial feasible solution, namely  $\mathbf{x} \neq \mathbf{0}$ . A nontrivial feasible solution satisfies the inequality  $\mathbf{e}^T \mathbf{x} > 0$ . In this case, the set of feasible solutions form a cone, thus there is a feasible solution which satisfies  $\mathbf{e}^T \mathbf{x} = 1$ . Therefore, solving (in polynomial time) the linear programming problem

$$(\mathbf{x}, \mathbf{s}) \in \mathcal{F}_P, \quad \mathbf{e}^T \mathbf{x} = 1,$$

we get a nontrivial feasible solution, which is a nontrivial solution of the  $(LCP)$  too. ■

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