McMaster University

Advanced Optimization Laboratory

Title:

A Redundant Klee-Minty Construction with All the Redundant Constraints Touching the Feasible Region

Authors:

Eissa Nematollahi and Tamás Terlaky

AdvOl-Report No. 2007/18

December 2007, Hamilton, Ontario, Canada

A Redundant Klee-Minty Construction with All the Redundant Constraints Touching the Feasible Region

Eissa Nematollahi Tamás Terlaky

January 7, 2008

Abstract

By introducing some redundant Klee-Minty constructions, we have previously shown that the central path may visit every vertex of the Klee-Minty cube having $2^n - 2$ "sharp" turns in dimension n. In all of the previous constructions, the maximum of the distances of the redundant constraints to the corresponding facets is an exponential number of the dimension n, and those distances are decaying geometrically. In this paper, we provide a new construction in which all of the distances are set to zero, i.e., all of the redundant constraints touch the feasible region.

Key words: Linear optimization, Klee-Minty cube, interior point methods, redundant, central path.

1 Introduction

The *simplex method* was introduced by Dantzig [1] in 1947 for solving linear optimization problems (LPs). In 1972, Klee and Minty [7] showed that the simplex method may take an exponential number of iterations to find an optimal solution. More precisely, they presented an LP over an n-dimensional squashed cube and proved that a variant of the simplex method visits all of its 2^n vertices before reaching the optimal solution. The pivot rule used in [7] was the most negative reduced cost pivot rule that is frequently referred to as "Dantzig rule". Variants of the Klee-Minty cube have been used to prove exponential running time for most pivot rules; see [12] and the references therein for details. Stimulated mostly by the Klee-Minty worst-case example, the search for a polynomial algorithm for solving LPs has started.

In 1979, Khachiyan [6] proved that the *ellipsoid method* solves LPs in polynomial time. In spite of its polynomial iteration complexity, the ellipsoid method turned out to be inefficient in computational practice. In 1984, Karmarkar [5] proposed a polynomial time algorithm with a better complexity bound that sparked the research on polynomial time *interior point methods (IPMs).* Unlike the simplex method that goes along the edges of the polytope corresponding to the feasible region, IPMs pass through the interior of this polytope. Starting at a neighborhood of the *analytic center*[1], most IPMs follow the so-called *central path*[1] and converge to the analytic center of the optimal face; see for example [10, 13]. It is well known that the number of iterations needed to have the duality gap smaller than ϵ is bounded above by $O\big(\sqrt{N}\ln\frac{v_0}{\epsilon}\big)$, where N and v_0 denote the number of inequalities and the duality gap at the starting point, respectively.

[1]See page 4 for the definition of the analytic center and the central path.

The standard rounding procedure [10] can be used to compute an exact optimal solution with the choice of $\epsilon \leq 2^{-O(L)}$, where L is the bit-length of the input data. In this case, the iteration complexity becomes $O(\sqrt{N}L)$.

In [2], the authors show that the central path of a redundant representation of the Klee-Minty cube may trace the *simplex path*–the edge-path followed by the simplex method. More precisely, an exponential number of redundant constraints parallel to the facets passing through the optimal vertex are added to the Klee-Minty cube to force the central path to visit an arbitrarily small neighborhood of each vertex of that cube, thus having $2^n - 2$ "sharp" turns. The distances of the redundant constraints to the corresponding facets are chosen to be uniform (and at least $d \geq n2^{n+1}$), and the number of the inequalities is required to be at least $N = O(n^2 2^{6n})$, which is further improved to $N = O(n2^{3n})$ in [4] by a meticulous analysis. In [3], those distances are allowed to decay geometrically as $d = n(2^{n+4}, \ldots, 2^{n-k+5}, \ldots, 2^5)^T$, and the number of the redundant constraints is significantly reduced to $N = O(n^3 2^{2n})$. A simplified construction, where the number of the redundant constraints is further reduced to $N = O(n2^{2n})$, is presented in [9] by placing the redundant constraints parallel to the coordinate hyperplanes at geometrically decaying distances $d \approx \left(2^{n-1}, \ldots, 2^{n-k}, \ldots, 2, 0\right)^T$.

In this paper, we present a new redundant Klee-Minty construction with $d = 0$. In other words, all of the redundant constraints touch the feasible region. The number of the redundant constraints is required to be $N = O\left(2^{n^2}\right)$, which is exponentially larger than those of the previous constructions.

The rest of the paper is organized as follows. In Section 2, we introduce our redundant Klee-Minty problems and present the main results as three propositions, whose proofs are provided in Section 3. We use the following notations. The largest integer smaller than a scalar α is denoted by $\lfloor \alpha \rfloor$. For any vector $x = (x_1, \ldots, x_n)^T$, the vector $(\lfloor x_1 \rfloor, \ldots, \lfloor x_n \rfloor)^T$ is denoted by $\lfloor x \rfloor$. The unique maximizer of a strictly concave function $f(x)$ over a convex set $S \subseteq \mathbb{R}^n$ is denoted by $\arg\max_{x \in S} f(x)$.

2 The Main Results

We consider the following Klee-Minty problem [7], with the convention $y_0 = 0$, where τ is a small positive factor by which the unit cube $[0,1]^n$ is squashed.

$$\begin{array}{lll} \min & y_n & \\ \text{subject to} & \tau\, y_{k-1} \leq y_k \leq 1 - \tau\, y_{k-1}, & \text{for } k = 1, \ldots, n. \end{array} \tag{1}$$

The polytope represented by all of the $2n$ inequalities of (1) is denoted by $\mathcal{C}_0^n$. Variants of the simplex method may take $2^n - 1$ iterations to solve this problem; see the survey paper [12] and the references therein. Starting from the vertex $(0, \ldots, 0, 1)^T$, they may visit all of the vertices of the polytope ordered by the decreasing values of the last coordinate y_n until reaching the optimal point, which is the origin.

We consider redundant constraints induced by the halfplanes $y_k \geq 0$ repeated h_k times, for $k = 1, \ldots, n$. Therefore, the problem that we are interested in is

$$\begin{array}{lll} \min & y_n & \\ \text{subject to} & \tau\, y_{k-1} \leq y_k \leq 1 - \tau\, y_{k-1}, & \text{for } k = 1, \ldots, n, \\ & 0 \leq y_k, & \text{repeated } h_k \text{ times, for } k = 1, \ldots, n. \end{array} \tag{2}$$

Let $h = (h_1, \ldots, h_n)^T$ be the repetition vector. We denote by $\mathcal{C}_h^n$ the polytope represented by all of the constraints of (2). Obviously, problem (1) is a special case of (2) with $h = 0$. Observe that all of the redundant constraints of (2) touch the feasible region.

Let $y = (y_1, \ldots, y_n)^T$. The *analytic center* of the polytope $\mathcal{C}_h^n$ is defined to be the unique solution of the strictly concave maximization problem

$$\chi^h = \arg\max_y \sum_{k=1}^{n} (\ln s_k + \ln \bar{s}_k + h_k \ln y_k),$$

where, for $k = 1, \ldots, n$,

$$s_k = y_k - \tau y_{k-1}, \quad \bar{s}_k = 1 - y_k - \tau y_{k-1}.$$

It can be seen that $\chi^h \neq \chi^0$ for $h \neq 0$ due to the fact that the polytopes represented by $\mathcal{C}_h^n$ and $\mathcal{C}_0^n$ coincide, although their actual algebraic representations are different.

Remark 2.1. To avoid confusion, we denote the identical feasible regions of (1) and (2) by $\mathcal{C}^n$, which is also well known as the Klee-Minty cube. Note that the analytic center of $\mathcal{C}^n$ cannot be defined without making its algebraic representation clear, while the analytic centers of $\mathcal{C}_h^n$ and $\mathcal{C}_0^n$ are well defined.

From the necessary and sufficient optimality conditions (the gradient is equal to zero), the analytic center χ^h is the unique solution of the system

$$\begin{cases} \frac{1}{s_k} - \frac{\tau}{s_{k+1}} - \frac{1}{\bar{s}_k} - \frac{\tau}{\bar{s}_{k+1}} + \frac{h_k}{y_k} = 0, & \text{for } k = 1, \ldots, n-1, \\ \frac{1}{s_n} - \frac{1}{\bar{s}_n} + \frac{h_n}{y_n} = 0, & \\ s_k > 0, \bar{s}_k > 0, y_k > 0, & \text{for } k = 1, \ldots, n. \end{cases} \tag{3}$$

The *central path* of (2) is defined to be the set

$$\mathcal{P}_h = \left\{ y(\mu) \mid y(\mu) = \arg\max_y \; -y_n + \mu \sum_{k=1}^{n} (\ln s_k + \ln \bar{s}_k + h_k \ln y_k), \quad \mu > 0 \right\}.$$

It is easy to see that any point $y(\mu)$ on the central path $\mathcal{P}_h$ satisfies all of the equations of (3), except the last one.

By analogy with the unit cube $[0,1]^n$, we denote the vertices of the Klee-Minty cube $\mathcal{C}^n$ by using subsets of $\{1, \ldots, n\}$. For $S \subseteq \{1, \ldots, n\}$, a vertex $v^S = (v_1^S, \ldots, v_n^S)^T$ of $\mathcal{C}^n$ is defined by

$$v_k^S \;=\; \begin{cases} 1 - \tau v_{k-1}^S, & \text{if } k \in S \\ \tau v_{k-1}^S, & \text{otherwise,} \end{cases} \qquad k = 1, \ldots, n,$$

with the convention that $v_0^S = 0$. The definition is illustrated by Figure 1. The δ-neighborhood of a vertex v^S is defined, see Figure 2, by

$$\mathcal{N}_\delta(v^S) = \left\{ y \in \mathcal{C}^n \mid \begin{cases} \bar{s}_k \le \delta, & \text{if } k \in S \\ s_k \le \delta, & \text{otherwise} \end{cases} \quad k = 1, \ldots, n \right\}.$$

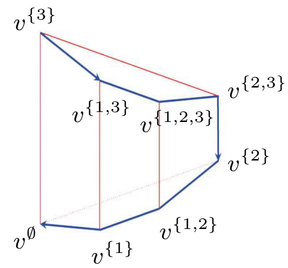

Figure 1: The vertices of $\mathcal{C}^3$ and the simplex path P_0.

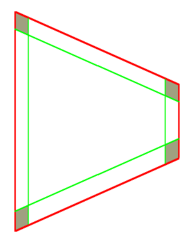

Figure 2: The δ-neighborhoods of the 4 vertices of $\mathcal{C}^2$.

To ease the analysis, we provide a mathematical definition for the simplex path and its δ-neighborhood in $\mathcal{C}^n$. For this purpose, we first define, for $k = 2, \ldots, n$, the sets

$$T_\delta^k = \{y \in \mathcal{C}^n \mid \bar{s}_k \le \delta\}, \quad C_\delta^k = \{y \in \mathcal{C}^n \mid \bar{s}_k > \delta, s_k > \delta\}, \quad B_\delta^k = \{y \in \mathcal{C}^n \mid s_k \le \delta\},$$

and the set $\hat{C}_\delta^k = \{y \in \mathcal{C}^n \mid \bar{s}_k \le \delta, s_{k-1} \le \delta, \ldots, s_1 \le \delta\}$, for $k = 1, \ldots, n$. Visually, the sets T_δ^k, C_δ^k, and B_δ^k can be considered as the top, central, and bottom parts of $\mathcal{C}^n$, and obviously $\mathcal{C}^n = T_\delta^k \cup C_\delta^k \cup B_\delta^k$, for $k = 1, \ldots, n$. Then, a δ-neighborhood of the simplex path, see Figure 3, might be given as $P_\delta = \bigcap_{k=2}^n A_\delta^k$, where $A_\delta^k = T_\delta^k \cup \hat{C}_\delta^{k-1} \cup B_\delta^k$, for $k = 2, \ldots, n$. The simplex path itself, see Figure 1, is precisely determined by $P_0 = \bigcap_{k=2}^n A_0^k$.

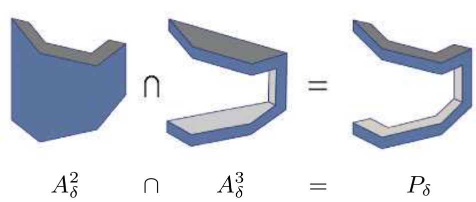

Figure 3: The set P_δ, the δ-neighborhood of the simplex path, for $\mathcal{C}^3$.

In the rest of this section, we focus on (2) with the following choice of parameters

$$\bar{\tau} = \frac{n}{2(n+1)}, \quad \bar{\delta} = \frac{1}{4(n+1)}, \quad \bar{h} = \left\lfloor \left(\frac{2}{\bar{\delta}}, \frac{4}{\bar{\tau}\bar{\delta}^2}, \ldots, \frac{2n}{\bar{\tau}^{n(n-1)/2}\bar{\delta}^n} \right)^T \right\rfloor.$$

The resulting redundant minimization problem, which depends only on n, is referred to as problem (RP^n). For the sake of simplicity, we denote $\mathcal{C}_{\bar{h}}^n$ of (RP^n) by $\bar{\mathcal{C}}^n$. Obviously, $\bar{\tau} + \bar{\delta} < 1/2$, so the $\bar{\delta}$-neighborhoods of the 2^n vertices of $\bar{\mathcal{C}}^n$ are non-overlapping. The analytic center of $\bar{\mathcal{C}}^n$ is denoted by χ^n, and the central path of (RP^n) is denoted by $\mathcal{P}^n$. The following proposition gives the number of inequalities of (RP^n). Its proof is given in Section 3.3.

Proposition 2.1. *The number of the inequalities in* (RP^n) *is* $N = O\big(2^{n^2}\big)$.

Proposition 2.2 is to ensure that the analytic center χ^n is in the $\bar{\delta}$-neighborhood of the vertex $v^{\{n\}}$, which is precisely $\hat{C}^n_{\bar{\delta}}$. The proof of the proposition is presented in Section 3.4.

Proposition 2.2. *The analytic center χ^n of $\bar{\mathcal{C}}^n$ is in the $\bar{\delta}$-neighborhood of $v^{\{n\}}$, i.e., $\chi^n \in \hat{C}^n_{\bar{\delta}}$.*

Proposition 2.3 states that the central path $\mathcal{P}^n$ of (RP^n) takes at least $2^n - 2$ turns before converging to the origin as it stays in the $\bar{\delta}$-neighborhood of the simplex path. Thus, the central path $\mathcal{P}^n$ visits the $\bar{\delta}$-neighborhoods of all of the 2^n vertices of $\mathcal{C}^n$. The proof of the proposition is presented in Section 3.5.

Proposition 2.3. *The central path $\mathcal{P}^n$ of* (RP^n) *stays in the $\bar{\delta}$-neighborhood of the simplex path of $\mathcal{C}^n$, i.e., $\mathcal{P}^n \subset P_{\bar{\delta}}$.*

3 The Proofs of Propositions 2.1, 2.2, and 2.3

3.1 Preliminary Results

We show that the central path $\mathcal{P}_h$ of (2) is bent along the simplex path of the Klee-Minty cube $\mathcal{C}^n$ so that it visits the δ-neighborhood of every vertex of that cube, providing that h satisfies

$$h_k \tau^{k-1} \delta \quad \geq \quad 1 + 2\tau^k + \sum_{i=1}^{k-1} h_i, \qquad \text{for } k = 1, \ldots, n, \tag{4}$$

where by convention $\sum_{i=1}^{0} h_i = 0$. For $k = 1, \ldots, n$, the k^{th} inequality of (4) ensures that the central path $\mathcal{P}_h$ is pushed enough toward the set $\hat{C}^k_\delta$. In the following lemma, we prove some implications of inequality system (4).

Lemma 3.1. *For any $k = 2, \ldots, n$, the k^{th} inequality of* (4) *implies all of the following inequalities*

$$h_k \tau^{k-1} \delta \quad \geq \quad 2\tau^k + \tau^{m-1}\Big(1 + \sum_{i=m}^{k-1} h_i\Big), \qquad \textit{for } m = 1, \ldots, k-1.$$

Proof. The proof immediately follows as $\tau^{m-1} < 1$ and $h_i > 0$ for $i = 1, \ldots, m-1$. □

We now present the main theorem of this section.

Theorem 3.2. *Let $\tau + \delta < 1/2$ and h satisfy* (4). *Then, for problem* (2), *we have*

$$\mathcal{P}_h \cap C^{k+1}_\delta \subseteq \hat{C}^k_\delta, \qquad \textit{for } k = 1, \ldots, n-1.$$

Proof. We show that for all $k = 1, \dots, n-1$, any point on the central path $\mathcal{P}_h$ that satisfies $s_{k+1} > \delta$ and $\bar{s}_{k+1} > \delta$, also satisfies

$$\bar{s}_k \leq \delta,\ s_{k-1} \leq \delta,\ \dots,\ s_1 \leq \delta.$$

Recall that any point on the central path $\mathcal{P}_h$ satisfies the first $n-1$ equations of (3). From the k^{th} equation of (3), we have

$$\frac{h_k}{y_k} = -\frac{1}{s_k} + \frac{1}{\bar{s}_k} + \frac{\tau}{s_{k+1}} + \frac{\tau}{\bar{s}_{k+1}},$$

which, since $s_{k+1} > \delta$ and $\bar{s}_{k+1} > \delta$, implies that

$$h_k \leq \frac{1}{\bar{s}_k} + \frac{2\tau}{\delta}.$$

Since by (4) the inequality $h_k \geq (1+2\tau)/\delta$ holds, we get $\bar{s}_k \leq \delta$.

Let $1 \leq m \leq k-1$. Adding the k^{th} equation of (3) multiplied by τ^{k-1} to the j^{th} equation of (3) multiplied by $-\tau^{j-1}$, for all $j = m, \dots, k-1$, we have

$$\frac{h_k \tau^{k-1}}{y_k} - \sum_{i=m}^{k-1} \frac{h_i \tau^{i-1}}{y_i} \quad \leq \quad \frac{\tau^{m-1}}{s_m} + \frac{2\tau^k}{\delta}.$$

Using the facts that $y_i \geq \tau^{i-m} y_m$, for any $i = m, \dots, n$, and $y_m \geq s_m$, we obtain

$$h_k \tau^{k-1} - \sum_{i=m}^{k-1} \frac{h_i \tau^{m-1}}{s_m} \quad \leq \quad \frac{\tau^{m-1}}{s_m} + \frac{2\tau^k}{\delta},$$

or equivalently

$$h_k \tau^{k-1} \quad \leq \quad \Big(1 + \sum_{i=m}^{k-1} h_k\Big) \frac{\tau^{m-1}}{s_m} + \frac{2\tau^k}{\delta},$$

which, using the inequalities of Lemma 3.1, implies that $s_m \leq \delta$. ∎

3.2 Existence of an Integer-valued Repetition Vector h

In this subsection, we show that there exists an integer-valued repetition vector h that satisfies (4). In the rest of this subsection, we assume that $\tau + \delta < 1/2$. Let us start with the following lemma which gives inequalities that are used in the subsequent lemmas.

Lemma 3.3. *For any $k = 1, 2, \dots,$ we have*

$$1 + 2\tau + \sum_{i=1}^{k} \frac{i}{\tau^{i(i-1)/2} \delta^i} \quad \leq \quad \frac{k+1}{\tau^{k(k-1)/2} \delta^k}.$$

Proof. The proof is by induction on k. The inequality is obviously true for $k = 1$ as $\tau + \delta < 1/2$. Assuming that the inequality holds for all $k = 1, \ldots, j$, we have

$$1 + 2\tau + \sum_{i=1}^{j+1} \frac{i}{\tau^{i(i-1)/2}\delta^i} \leq \frac{j+1}{\tau^{j(j-1)/2}\delta^j} + \frac{j+1}{\tau^{j(j+1)/2}\delta^{j+1}} = \frac{(j+1)(1+\tau^j\delta)}{\tau^{j(j+1)/2}\delta^{j+1}} \leq \frac{j+2}{\tau^{j(j+1)/2}\delta^{j+1}}.$$

Therefore, the inequality holds for $j = k + 1$, and the proof is completed. ∎

The following lemma provides an explicit solution of (4).

Lemma 3.4. *The vector* $h = \left(\frac{1}{\delta}, \frac{2}{\tau\delta^2}, \ldots, \frac{n}{\tau^{n(n-1)/2}\delta^n}\right)^T$ *satisfies* (4).

Proof. Obviously $h_1 \geq 1/\delta$. Therefore, it suffices to show that, for $k = 2, \ldots, n$,

$$\frac{k\tau^{k-1}\delta}{\tau^{k(k-1)/2}\delta^k} \quad \geq \quad 1 + 2\tau + \sum_{i=1}^{k-1} \frac{i}{\tau^{i(i-1)/2}\delta^i}.$$

This inequality is equivalent to

$$\frac{k}{\tau^{(k-1)(k-2)/2}\delta^{k-1}} \quad \geq \quad 1 + 2\tau + \sum_{i=1}^{k-1} \frac{i}{\tau^{i(i-1)/2}\delta^i},$$

which hold as shown in Lemma 3.3. ∎

We look for an integer-valued solution $\bar{h}$ of (4), since every component of h represents the repetition number of the corresponding coordinate-plane. The following lemma shows how to construct an integer-valued solution from an arbitrary solution h of (4).

Lemma 3.5. *If h is a solution of* (4)*, then so is the integer-valued vector* $\bar{h} = \lfloor 2h \rfloor$.

Proof. For any $k = 1, \ldots, n$, there exists $0 \leq \varepsilon_k < 1$ such that $\bar{h}_k = 2h_k - \varepsilon_k$. By assumption, h satisfies (4). Therefore, multiplying each side of (4) by 2 and substituting $2h_k$ by $\bar{h}_k + \varepsilon_k$, for all $k = 1, \ldots, n$, we get

$$\bar{h}_k\tau^{k-1} + \varepsilon_k\tau^{k-1}\delta \quad \geq \quad 2 + 4\tau^k + \sum_{i=1}^{k-1} \bar{h}_i + \sum_{i=1}^{k-1} \varepsilon_i,$$

implying

$$\bar{h}_k\tau^{k-1}\delta \geq 2 + 4\tau^k - \varepsilon_k\tau^{k-1}\delta + \sum_{i=1}^{k-1} \bar{h}_i + \sum_{i=1}^{k-1} \varepsilon_i \geq 2 + 4\tau^k - \delta + \sum_{i=1}^{k-1} \bar{h}_i \geq 1 + 2\tau^k + \sum_{i=1}^{k-1} \bar{h}_i.$$

Therefore, the integer-valued vector $\bar{h}$ satisfies (4). ∎

3.3 Proof of Proposition 2.1

The number of the inequalities of (RP^n) is

$$\begin{aligned} N &= 2n + \sum_{k=1}^{n} \bar{h}_k, \\ &\leq 2n + \sum_{k=1}^{n} 2k\Big(\frac{2n+2}{n}\Big)^{k(k-1)/2} (4n+4)^k, \\ &\leq 2n + 2n(n+1)^n 2^{2n}\Big(1+\frac{1}{n}\Big)^{n(n-1)/2} \sum_{k=1}^{n} 2^{k(k-1)/2}. \end{aligned}$$

Since $(1+1/n)^n \leq e$ and $\sum_{k=1}^{n} 2^{k(k-1)/2} \leq 2^{n^2/2}$, we get

$$N \leq 2n + (n+1)^{n+1} 2^{2n+1} e^{(n-1)/2} 2^{n^2/2},$$

implying that $N = O\big(2^{n^2}\big)$. ∎

3.4 Proof of Proposition 2.2

The proof is similar to the proof of Theorem 3.2. The analytic center χ^n of $\bar{\mathcal{C}}^n$ is the unique solution of (3). We prove that any point that satisfies (3) also satisfies

$$\bar{s}_n \leq \bar{\delta},\ s_{n-1} \leq \bar{\delta},\ \ldots,\ s_1 \leq \bar{\delta}.$$

From Lemmas 3.4 and 3.5, the vector $\bar{h}$ satisfies all of the inequalities of (4). From the n^{th} equation of (3), we have

$$\frac{\bar{h}_n}{y_n} \leq \frac{1}{\bar{s}_n}.$$

Since $\bar{h}_n \geq 1/\bar{\delta}$ holds from (4), we get $\bar{s}_n \leq \bar{\delta}$.

Let $1 \leq m \leq n-1$. Adding the n^{th} equation of (3) multiplied by $\bar{\tau}^{n-1}$ to the j^{th} equation of (3) multiplied by $-\bar{\tau}^{j-1}$, for all $j = m \ldots, n-1$, we have

$$\frac{\bar{h}_n \bar{\tau}^{n-1}}{y_n} - \sum_{i=m}^{n-1} \frac{\bar{h}_i \bar{\tau}^{i-1}}{y_i} \leq \frac{\bar{\tau}^{m-1}}{s_m} + \frac{2\bar{\tau}^n}{\bar{\delta}}.$$

Using the facts that $y_i \geq \bar{\tau}^{i-m} y_m$, for any $i = m, \ldots, n$, and $y_m \geq s_m$, we obtain

$$\bar{h}_n \bar{\tau}^{n-1} - \sum_{i=m}^{n-1} \frac{\bar{h}_i \bar{\tau}^{m-1}}{s_m} \leq \frac{\bar{\tau}^{m-1}}{s_m} + \frac{2\bar{\tau}^n}{\bar{\delta}},$$

or equivalently

$$\bar{h}_n \bar{\tau}^{n-1} \leq \Big(1 + \sum_{i=m}^{n-1} \bar{h}_n\Big) \frac{\bar{\tau}^{m-1}}{s_m} + \frac{2\bar{\tau}^n}{\bar{\delta}},$$

which, using the inequalities of Lemma 3.1, implies that $s_m \leq \bar{\delta}$. ∎

3.5 Proof of Proposition 2.3

We show that the central path $\mathcal{P}^n$ of (RP^n) is contained in the $\bar{\delta}$-neighborhood of the simplex path $P_{\bar{\delta}} = \bigcap_{k=2}^n A_{\bar{\delta}}^k$. By Proposition 2.2, the starting point χ^n of $\mathcal{P}^n$, which is the analytic center of $\bar{\mathcal{C}}^n$, belongs to $\hat{C}_{\bar{\delta}}^n = \mathcal{N}_{\bar{\delta}}\left(v^{\{n\}}\right)$. Since $\mathcal{C}^n = \bigcap_{k=2}^n (T_{\bar{\delta}}^k \cup C_{\bar{\delta}}^k \cup B_{\bar{\delta}}^k)$, we have

$$\mathcal{P}^n = \bigcap_{k=2}^{n} (T_{\bar{\delta}}^k \cup C_{\bar{\delta}}^k \cup B_{\bar{\delta}}^k) \cap \mathcal{P}^n = \bigcap_{k=2}^{n} (T_{\bar{\delta}}^k \cup (C_{\bar{\delta}}^k \cap \mathcal{P}^n) \cup B_{\bar{\delta}}^k) \cap \mathcal{P}^n.$$

By Theorem 3.2, $\mathcal{P}^n \subset \bigcap_{k=2}^n (T_{\bar{\delta}}^k \cup \hat{C}_{\bar{\delta}}^{k-1} \cup B_{\bar{\delta}}^k) = \bigcap_{k=2}^n A_{\bar{\delta}}^k = P_{\bar{\delta}}$. ∎

Acknowledgments. Research was supported by the NSERC Discovery grant #48923 and a MITACS grant for both authors and by the Canada Research Chair program for the second author.

References

[1] G. B. Dantzig, *Maximization of a linear function of variables subject to linear inequalities*, in Activity Analysis of Production and Allocation, T. C. Koopmans, ed., John Wiley, 1951, pp. 339–347.

[2] A. Deza, E. Nematollahi, R. Peyghami, and T. Terlaky, *The central path visits all the vertices of the Klee-Minty cube*, Optimization Methods and Software, 21 (2006), pp. 851–865.

[3] A. Deza, E. Nematollahi, and T. Terlaky, *How good are interior-point methods? Klee-Minty cubes tighten iteration-complexity bounds*, Mathematical Programming. To appear. http://www.springerlink.com/content/d14343531t7404t0/.

[4] A. Deza, T. Terlaky, and Y. Zinchenko, *Central path curvature and iteration-complexity for redundant Klee-Minty cubes*, in Complementarity, Duality, and Global Optimization, D. Gao and H. Sherali, eds., Springer, 2006, pp. 217–249.

[5] N. K. Karmarkar, *A new polynomial-time algorithm for linear programming*, Combinatorica, 4 (1984), pp. 373–395.

[6] L. G. Khachiyan, *A polynomial algorithm in linear programming*, Soviet Mathematics Doklady, 20 (1979), pp. 191–194.

[7] V. Klee and G. J. Minty, *How good is the simplex algorithm?*, in Inequalities III, O. Shisha, ed., Academic Press, 1972, pp. 159–175.

[8] N. Megiddo, *Pathways to the optimal set in linear programming*, in Progress in Mathematical Programming: Interior-Point and Related Methods, N. Megiddo, ed., Springer-Verlag, 1988, pp. 131–158. Also In Proceedings of the 7th Mathematical Programming Symposium of Japan, 1–35, Nagoya, Japan, 1986.

[9] E. Nematollahi and T. Terlaky, *A simpler and tighter redundant Klee-Minty construction*, Optimization Letters. To appear.
`http://www.springerlink.com/content/x013qq7g16kg3347/`.

[10] C. Roos, T. Terlaky, and J.-P. Vial, *Theory and Algorithms for Linear Optimization: An Interior Point Approach*, Springer, New York, NY, second ed., 2006.

[11] G. Sonnevend, *An "analytical centre" for polyhedrons and new classes of global algorithms for linear (smooth, convex) programming*, Lecture Notes in Control and Information Sciences, 84 (1986), pp. 866–876.

[12] T. Terlaky and S. Zhang, *Pivot rules for linear programming – a survey*, Annals of Operations Research, 46 (1993), pp. 203–233.

[13] Y. Ye, *Interior-Point Algorithms: Theory and Analysis*, Wiley-Interscience Series in Discrete Mathematics and Optimization, John Wiley, 1997.

Eissa Nematollahi and Tamás Terlaky
Advanced Optimization Laboratory, Department of Computing and Software,
School of Computational Engineering and Science,
McMaster University, Hamilton, Ontario, Canada.
Email: nematoe, terlaky@mcmaster.ca