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Abstract

A partial Steiner (k, l)-system is a k-uniform hypergraph $\mathcal{G}$ with the property that every l-element subset of V is contained in at most one edge of $\mathcal{G}$. In this paper we show that for given k, l and t there exists a partial Steiner (k, l)-system such that whenever an l-element subset from every edge is chosen, the resulting l-uniform hypergraph contains a clique of size t. As the main result of this note, we establish asymptotic lower and upper bounds on the size of such cliques with respect to the order of Steiner systems.

Keywords: Steiner system, hypergraph, clique.

1 Introduction

A *partial Steiner (k,l)-system* ((k, l)-system in short) is a k-uniform hypergraph $\mathcal{G} = (V, \mathcal{E})$ with the property that every l-element subset of V is contained in at most one edge of $\mathcal{G}$. For fixed k and l we denote the set of all (k, l)-systems by $S(k, l)$. Questions regarding the maximum numbers of edges in (k, l)-systems have been studied, *e.g.*, in [1, 8, 17]. Another direction of the research was pioneered by Alex Rosa [15, 16], who was the first to investigate questions regarding the chromatic number of Steiner systems. This motivated a further study on chromatic numbers and independent sets of Steiner systems by a number of researchers (see, *e.g.*, [4, 5, 6, 9, 12, 18]). The aim of this note is to introduce a Ramsey type parameter related to (k, l)-systems. The following notion of a selector is essential for the discussion. Let $l < k$ be integers and $\mathcal{H} = (V, \mathcal{E})$ be a k-uniform hypergraph. A *selector* is a function $\mathsf{S} : \mathcal{E} \to [V]^l$ satisfying $\mathsf{S}(E) \subset E$ for every $E \in \mathcal{E}$. Moreover, denote by $K_n^{(l)}$ the l-uniform complete hypergraph of order n. We can reformulate the well-known Ramsey theorem (see, *e.g.*, [10, 14]) using this notion to:

Theorem 1.1. *Let k, l, t be integers satisfying $l \le \min\{k, t\}$. Then, there exists n such that the hypergraph $K_n^{(k)} = (V, \mathcal{E})$ has the following property. For any selector $\mathsf{S} : \mathcal{E} \to [V]^l$ the l-uniform hypergraph $\big(V, \mathsf{S}(\mathcal{E})\big)$ contains a clique $K_t^{(l)}$.*

Note that the smallest such integer equals $R_l(k, t)$. In this note we are interested in an extension of Theorem 1.1 with $\big(V, \binom{V}{k}\big)$ replaced by a "sparse" hypergraph. Clearly, if $(V, \mathcal{E})$ is a partial $(k, l-1)$-system, then for any selector S and $E, E' \in \mathcal{E}$, $E \neq E'$, $|\mathsf{S}(E) \cap \mathsf{S}(E')| \le l-2$ holds, and consequently, the l-uniform hypergraph $\big(V, \mathsf{S}(\mathcal{E})\big)$ cannot contain a clique $K_t^{(l)}$. We show, however, that with a conveniently chosen (k, l)-system Theorem 1.1 remains true.

Theorem 1.2. *Let k, l, t be integers satisfying $l \le \min\{k, t\}$. Then, there exists a (k, l)-system $\mathcal{T} = (V, \mathcal{E})$ such that for any selector $\mathsf{S} : \mathcal{E} \to [V]^l$ the l-uniform hypergraph $\big(V, \mathsf{S}(\mathcal{E})\big)$ contains a clique $K_t^{(l)}$.*

The special case of Theorem 1.2 (for $k = 3$ and $l = 2$) follows from the result of the second author [7], where it was shown that for any positive integer t and n large enough every *projective Steiner triple system* $PG(n, 2)$ (*cf.* [3]) satisfies the conditions of Theorem 1.2. In other words (for $k = 3$ and $l = 2$), projective Steiner triple systems $PG(n, 2)$ have the property of $(3, 2)$-system $\mathcal{T}$ with n sufficiently large.

Theorem 1.2, though quite powerful, gives no explicit estimate of the size of the graph $\mathcal{T}$. A quantitative extension of Theorem 1.2 is the main result of this note.

Let $\mathcal{H}$ be an l-uniform hypergraph. Define the *clique number* of $\mathcal{H}$ as

$$\omega(\mathcal{H}) = \max\big\{t \in \mathbb{N} \mid \mathcal{H} \supseteq K_t^{(l)}\big\}.$$

Let $\mathcal{G} = (V, \mathcal{E})$ be (k, l)-system. Define also the clique number for (k, l)-system as

$$\omega(\mathcal{G}, k, l) = \min\big\{\omega\big((V, \mathsf{S}(\mathcal{E}))\big) \mid \mathsf{S} \text{ is a selector on } \mathcal{G}\big\}.$$

Furthermore, let

$$\omega(n, k, l) = \max\big\{\omega(\mathcal{G}) \mid \mathcal{G} \text{ is } (k, l)\text{-system of order } n\big\}.$$

Theorem 1.2 states that for any fixed k and l the function $\omega(n, k, l) \to \infty$ as $n \to \infty$. For *partial Steiner triple systems* (PSTS), *i.e.*, where $k = 3$ and $l = 2$, we show the following explicit bounds.

Theorem 1.3.

$$(1 - o(1)) \log_2 \log_2 n \le \omega(n, 3, 2) \le 2 \log_3 n + 1.$$

2 Proof of Theorem 1.2

Let $\le$ be a linear ordering of vertices V. For a given hypergraph $\mathcal{G} = (V, \mathcal{E})$ denote by $(\mathcal{G}, \le)$ the hypergraph with linear ordering $\le$ on its vertices. Let $(\mathcal{G}, \le)$ and $(\mathcal{H}, \le)$ be two ordered hypergraphs with $\mathcal{G} = (V, \mathcal{E})$ and $\mathcal{H} = (W, \mathcal{F})$. Say the mapping $\phi : V \to W$ is an *ordered embedding* if for all $v < v'$, $v, v' \in V$, $\phi(v) < \phi(v')$, and $\{\phi(v_1), \phi(v_2), \dots, \phi(v_k)\} \in \mathcal{F}$ if and only if $\{v_1, v_2, \dots, v_k\} \in \mathcal{E}$.

We use the Ramsey theorem for Steiner systems established by J. Nešetřil and the third author.

Theorem 2.1 ([13]). *Let $(\mathcal{G}, \le)$ be an ordered k-uniform hypergraph such that $\mathcal{G} \in S(k, l)$. Let $r \ge 2$ be an integer. Then, there exists an ordered k-uniform hypergraph $(\mathcal{H}, \le)$ with $\mathcal{H} \in S(k, l)$ and such that for every partition of the edges $\mathcal{E}(\mathcal{H}) = \mathcal{E}_1 \cup \mathcal{E}_2 \cup \dots \cup \mathcal{E}_r$ there exists i, $1 \le i \le r$, and an ordered embedding $\phi : V(\mathcal{G}) \to V\big((V, \mathcal{E}_i)\big)$.*

Proof of Theorem 1.2. First we are going to define an ordered Steiner system $(\mathcal{G}, \le)$ to which we will apply Theorem 2.1. Let $[k] = \{1, \dots, k\}$. For each $L \in \binom{[k]}{l}$ consider an ordered set T_L, $|T_L| = t$ so that for $L \neq L'$, $T_L \cap T_{L'} = \emptyset$. Now for each T_L, where say $L = \{m_1 < \dots < m_l\}$, and every l-element subset $U = \{u_1 < \dots < u_l\} \subset T_L$ consider a k-element set $V(L, U) = \{v_1 < \dots < v_k\}$ such that:

(i) $v_{m_i} = u_i$ for each $i = 1, \dots, l$, and

(ii) $V(L, U) \cap V(L, U') \subset T_L$ for each L, and

(iii) $V(L, U) \cap V(L', U') = \emptyset$, whenever $U \neq U'$.

Observe that (ii) and (iii) is equivalent to saying that the sets $V(L, U) \setminus U$ are pairwise disjoint for distinct U and L. For each $L \in \binom{[k]}{l}$ set

$$V_L = \bigcup \big\{V(L, U) \mid U \in \tbinom{T_L}{l}\big\}$$

and

$$V = \bigcup \left\{ V_L \mid L \in \binom{[k]}{l} \right\}.$$

Let

$$\mathcal{E}_L = \left\{ V(L, U) \mid U \in \binom{T_L}{l} \right\}, \tag{1}$$

and

$$\mathcal{E} = \bigcup \left\{ \mathcal{E}_L \mid L \in \binom{[k]}{l} \right\}. \tag{2}$$

Clearly $\mathcal{G} = (V, \mathcal{E})$ is (k, l)-system. Let $\leq$ be an arbitrary linear extension of the order we considered on elements of V. Let $r = \binom{k}{l}$ and let $(\mathcal{H}, \leq)$ be a graph guaranteed by Theorem 2.1. We claim that $\mathcal{T}$ is the desired graph $\mathcal{H}$.

Consider an arbitrary selector $\mathsf{S} : \mathcal{H} \to \binom{V(\mathcal{H})}{l}$ (for convenience we identify $\mathcal{H}$ with its edge set). Consider the following partition of the edges of $\mathcal{H}$ as

$$\mathcal{H} = \bigcup \left\{ \mathcal{H}_L \mid L \in \binom{[k]}{l} \right\},$$

where $\mathcal{H}_L = \left\{ E \in \mathcal{H} \;\middle|\; E = \{x_1 < \cdots < x_k\} \text{ and } \mathsf{S}(E) = \{x_i \mid i \in L\} \right\}$. By Theorem 2.1 there exists $L_0 \in \binom{[k]}{l}$ and a copy of $(\mathcal{G}, \leq)$ in $(\mathcal{H}, \leq)$, say $(\mathcal{G}_0, \leq)$, such that $E(\mathcal{G}_0) \subset \mathcal{H}_{L_0}$. In particular, all edges $E = \{x_1 < \cdots < x_k\}$ of $\mathcal{E}_{L_0} \subset E(\mathcal{G}_0)$ (*cf.* (1) and (2)) have the property that $\mathsf{S}(E) = \{x_i \mid i \in L_0\}$. Consequently, the set $T_{L_0} = \bigcup \left\{ \mathsf{S}(E) \mid E \in \mathcal{E}_{L_0} \right\}$ induces a clique $K_t^{(l)}$.
□

3 Proof of Theorem 1.3

First, we find an upper bound on $\omega(n, 3, 2)$ by using a simple probabilistic argument.

Proof of Theorem 1.3 (upper bound). We show that for sufficiently large n the following inequality holds:

$$\omega(n, 3, 2) \leq 2 \log_3 n + 1. \tag{3}$$

In order to prove (3), it suffices to show that $\omega(\mathcal{G}, 3, 2) \leq 2 \log_3 n + 1$ for any PSTS $\mathcal{G}$ of order n. For a given PSTS $\mathcal{G} = (V, \mathcal{E})$ with $|V| = n$ we show that there exists a selector $\mathsf{S} : \mathcal{E} \to [V]^2$ for which $(M, \mathsf{S}(\mathcal{E}) \cap [M]^2)$ is not a complete graph, *i.e.*, $\mathsf{S}(\mathcal{E}) \cap [M]^2 \neq [M]^2$, whenever $M \subseteq V$ and $|M| > 2 \log_3 n + 1$. Let $\mathbb{S} : \mathcal{E} \to [V]^2$ be a *random selector* defined by $\Pr(\mathbb{S} = \mathsf{S}) = \frac{1}{\binom{3}{2}^{|\mathcal{E}|}}$ for every possible selector S on $\mathcal{G}$. For a fixed set M with $|M| = m$ we have

$$\Pr\left(\mathbb{S}(\mathcal{E}) \cap [M]^2 = [M]^2 \right) \leq 3^{-\binom{m}{2}},$$

since from $3^{|\mathcal{E}|}$ selectors at most $3^{|\mathcal{E}| - \binom{m}{2}}$ of them keep $[M]^2$ complete. Thus,

$$\Pr\left(\exists M \in [V]^m \text{ such that } \mathbb{S}(\mathcal{E}) \cap [M]^2 = [M]^2 \right) \leq \binom{n}{m} 3^{-\binom{m}{2}},$$

and equivalently

$$\Pr\left(\forall M \in [V]^m \text{ we have } \mathbb{S}(\mathcal{E}) \cap [M]^2 \neq [M]^2 \right) \geq 1 - \binom{n}{m} 3^{-\binom{m}{2}}. \tag{4}$$

Note that for $m > 2\log_3 n + 1$ we get $\binom{n}{m} \le n^m < \left(3^{\frac{m-1}{2}}\right)^m = 3^{\binom{m}{2}}$. Consequently, the right side of (4) is positive, *i.e.*, there exists a selector with the required property.
□

In order to prove the lower bound on $\omega(n, 3, 2)$ we need to show the existence of the PSTS with the property that any selector chooses a clique of size $\Omega\big(\ln\ln n\big)$. To this end, we construct a PSTS with the property that any sufficiently large subset of its vertices induces many triples. We need one auxiliary result, *i.e.*, Proposition 3.3, which follows from a special version of Lovász Local Lemma, *i.e.*, Corollary 3.2. Let $A_1, \dots, A_n$ be events in a probability space. A graph $\Gamma = (V, E)$ on the set vertices $\{1, 2, \dots, n\}$ is called a *dependency graph* for the events $A_1, \dots, A_n$ if for each i, $1 \le i \le n$, the event A_i is mutually independent of all the events $\{A_j \mid \{i, j\} \notin E\}$.

Lemma 3.1 (Lovász Local Lemma (see, *e.g.*, [2])). *Suppose that $\Gamma = (V, E)$ is a dependency graph for the events $A_1, \dots, A_n$ and suppose there are real numbers $x_1, \dots, x_n$ such that $0 < x_i < 1$ and $\Pr(A_i) \le x_i \prod_{\{i,j\}\in E}(1 - x_j)$ for all $1 \le i \le n$. Then, $\Pr\left(\bigcap_{i=1}^n \bar{A}_i\right) > 0$, i.e., with positive probability no event A_i holds.*

In the proof of Proposition 3.3 we will use the following consequence of Lemma 3.1.

Corollary 3.2. *(For a similar result see [19].) Let $A_1, \dots, A_n$ be events in a probability space with a dependency graph $\Gamma = (V, E)$. Suppose, there exist real numbers $y_1, \dots, y_n, \delta$ such that $0 < \delta < 1$, $0 < y_i \Pr(A_i) \le \delta$ and $\sum_{\{i,j\}\in E} y_j \Pr(A_j) \le (1 - \delta)\ln(y_i)$ for all $1 \le i \le n$. Then, $\Pr\left(\bigcap_{i=1}^n \bar{A}_i\right) > 0$.*

Proof. For each i, $1 \le i \le n$, set $x_i = y_i \Pr(A_i)$. Note that $0 < x_i < 1$ and

$$\begin{aligned}\prod_{\{i,j\}\in E}(1 - x_j) &= \prod_{\{i,j\}\in E}\Big(1 - y_j \Pr(A_j)\Big) \ge \prod_{\{i,j\}\in E}\exp\left(\frac{-y_j \Pr(A_j)}{1 - y_j \Pr(A_j)}\right)\\ &= \exp\left(-\sum_{\{i,j\}\in E}\frac{y_j \Pr(A_j)}{1 - y_j \Pr(A_j)}\right) \ge \exp\left(-\sum_{\{i,j\}\in E}\frac{y_j \Pr(A_j)}{1-\delta}\right)\\ &\ge \exp\big(-\ln(y_i)\big) = \frac{1}{y_i} = \frac{\Pr(A_i)}{x_i},\end{aligned}$$

and hence, the assumptions of Lemma 3.1 are satisfied.
□

In the following, e denotes the base of the natural logarithmic function.

Proposition 3.3. *There exists a positive constant c such that for any $\varepsilon > 0$ and any sufficiently large $n \ge n_0(\varepsilon)$ there exists a PSTS $\mathcal{G} = (V, \mathcal{E})$ with $|V| = n$ and with the property that whenever $M \subseteq V$ with $|M| = m \ge n^{\frac{1}{2}+\varepsilon}$, then $|\mathcal{E} \cap [M]^3| > \frac{c}{2n}\binom{m}{3}$.*

Proof. By a standard averaging argument it is enough to show that the statement holds for $m = \lceil n^{\frac{1}{2}+\varepsilon}\rceil$. Set $c = \frac{1}{102}$ and $\varepsilon > 0$ be given. Let $\mathcal{G} = (V, \mathbb{E})$ be a random 3-uniform hypergraph with vertex set V, $|V| = n$, and with hyperedges chosen independently with probability $p = \frac{c}{n}$. For $L \in [V]^4$ let A_L be the event that $|\mathbb{E} \cap [L]^3| \ge 2$, *i.e.*, there is a pair, which is contained in at least two triples. For $M \in [V]^m$ let B_M be the event that $|\mathbb{E} \cap [M]^3| \le \frac{p}{2}\binom{m}{3}$. Note that for $L, \hat{L} \in [V]^4$, A_L is independent of all $A_{\hat{L}}$ with $|L \cap \hat{L}| < 3$. Similarly, for $M \in [V]^m$, B_M is independent of all A_L and $B_{\hat{M}}$ with $|M \cap L| < 3$ and $|M \cap \hat{M}| < 3$, respectively. Let

$$A = \bigcap_{L\in[V]^4} \bar{A}_L \cap \bigcap_{M\in[V]^m} \bar{B}_M.$$

If $\Pr(A) > 0$, then there is a 3-uniform hypergraph $\mathcal{G} = (V, \mathcal{E})$ which is a PSTS (no pair is covered more than once) such that for any $M \subseteq V$ we have $|\mathcal{E} \cap [M]^3| > \frac{p}{2}\binom{m}{3}$.

In order to complete the proof, we show that $\Pr(A) > 0$. According to Corollary 3.2 (applied with $\delta = \frac{1}{100}$), it suffices to find positive real numbers y_L and z_M, for all $L \in [V]^4$ and $M \in [V]^m$, so that

$$y_L \Pr(A_L) \le \frac{1}{100}, \tag{5}$$

$$z_M \Pr(B_M) \le \frac{1}{100}, \tag{6}$$

$$\sum_{|\hat{L} \cap L| \ge 3} y_{\hat{L}} \Pr(A_{\hat{L}}) \quad + \sum_{|M \cap L| \ge 3} z_M \Pr(B_M) \le \frac{99}{100} \ln(y_L), \tag{7}$$

and

$$\sum_{|L \cap M| \ge 3} y_L \Pr(A_L) \quad + \sum_{|\hat{M} \cap M| \ge 3} z_{\hat{M}} \Pr(B_{\hat{M}}) \le \frac{99}{100} \ln(z_M). \tag{8}$$

First, let us estimate $\Pr(A_L)$ and $\Pr(B_M)$. For each $L \in [L]^4$

$$\Pr(A_L) = \binom{4}{2} p^2 (1-p)^2 + \binom{4}{3} p^3 (1-p) + p^4 < 6p^2,$$

since $p \le 1$. Hence, for every $L \in [V]^4$ we have

$$\Pr(A_L) \le \frac{6c^2}{n^2}. \tag{9}$$

To estimate $\Pr(B_M)$ we will use Chernoff's inequality (see, *e.g.*, Theorem 2.1 in [11]). Let $X \sim B\big(\binom{m}{3}, p\big)$ be a random variable with binomial distribution. Then, $E[X] = \binom{m}{3} p$ and Chernoff's inequality yields

$$\Pr(B_M) = \Pr\left(X \le \frac{1}{2} E[X]\right) \le \exp\left(-\frac{1}{8} E[X]\right) = \exp\left(-\frac{1}{8}\frac{c}{n}\binom{m}{3}\right).$$

Hence, for sufficiently large n,

$$\Pr(B_M) < \exp\left(-\frac{c}{50}\frac{m^3}{n}\right). \tag{10}$$

Now for every $L \in [V]^4$ define

$$y_L = 1 + \frac{1}{n},$$

and for every $M \in [V]^m$ define

$$z_M = \exp\left(\frac{c}{100}\frac{m^3}{n}\right).$$

For a given $L \in [V]^4$ and n large enough, we obtain by (9)

$$y_L \Pr(A_L) \le \left(1 + \frac{1}{n}\right)\frac{6c^2}{n^2} \le \frac{1}{100}.$$

Similarly, since $m = \lceil n^{\frac{1}{2}+\varepsilon} \rceil$, (10) yields for n large enough

$$z_M \Pr(B_M) \le \exp\left(\frac{c}{100}\frac{m^3}{n}\right)\exp\left(-\frac{c}{50}\frac{m^3}{n}\right) = \exp\left(-\frac{c}{100}\frac{m^3}{n}\right) \le \frac{1}{100}.$$

Thus, conditions (5) and (6) are satisfied. To complete the proof of Proposition 3.3 we need to show that conditions (7) and (8) are satisfied as well.

For a given $L \in [V]^4$ the number of $\hat{L}$'s such that $\hat{L} \in [V]^4$ and $|\hat{L} \cap L| \ge 3$ is $\binom{4}{3}(n-4) < 4n$, and the number of M's such that $M \in [V]^m$ and $|L \cap M| \ge 3$ is trivially less than $\binom{n}{m} \le \left(\frac{ne}{m}\right)^m$. Thus,

$$\begin{aligned}\sum_{|\hat{L}\cap L|\ge 3} y_{\hat{L}} \Pr(A_{\hat{L}}) \quad &+ \sum_{|M\cap L|\ge 3} z_M \Pr(B_M) \\ &\le 4n\left(1+\frac{1}{n}\right)\frac{6c^2}{n^2} + \left(\frac{ne}{m}\right)^m \exp\left(\frac{c}{100}\frac{m^3}{n}\right)\exp\left(-\frac{c}{50}\frac{m^3}{n}\right) \\ &= \left(1+\frac{1}{n}\right)\frac{24c^2}{n} + \exp\left(m\Big(-\frac{c}{100}\frac{m^2}{n} + \ln\left(\frac{ne}{m}\right)\Big)\right). \qquad (11)\end{aligned}$$

Since $m = \lceil n^{\frac{1}{2}+\varepsilon} \rceil$, then for sufficiently large n we have $-\frac{c}{100}\frac{m^2}{n} + \ln\left(\frac{ne}{m}\right) \le -1$. Hence, the second term of (11) can be estimated by

$$\exp\left(m\Big(-\frac{c}{100}\frac{m^2}{n} + \ln\left(\frac{ne}{m}\right)\Big)\right) \le \exp(-m) \le \exp(-\sqrt{n}),$$

which yields in (11) for sufficiently large n

$$\begin{aligned}&\left(1+\frac{1}{n}\right)\frac{24c^2}{n} + \exp\left(m\Big(-\frac{c}{100}\frac{m^2}{n} + \ln\left(\frac{ne}{m}\right)\Big)\right) \\ &\qquad \le \left(1+\frac{1}{n}\right)\frac{24c^2}{n} + \exp(-\sqrt{n}) = \frac{99}{100}\frac{2400c^2}{99}\left(1+\frac{1}{n}\right)\frac{1}{n} + \exp(-\sqrt{n}). \qquad (12)\end{aligned}$$

One can check that for $a < 1$ and x positive and sufficiently small number $a(1+x)x < \ln(1+x)$. Applying this inequality with $a = \frac{2400c^2}{99} < 1$ (recall $c = \frac{1}{102}$) yields that the right side of (12) can be bounded from above (for n large enough) by $\frac{99}{100}\ln\left(1+\frac{1}{n}\right)$. Thus,

$$\sum_{|\hat{L}\cap L|\ge 3} y_{\hat{L}} \Pr(A_{\hat{L}}) \quad + \sum_{|M\cap L|\ge 3} z_M \Pr(B_M) \;\le\; (11) \;\le\; (12) \;\le\; \frac{99}{100}\ln(y_L),$$

which proves (7).

Similarly, we show that (8) also holds. For a given $M \in [V]^m$, the number of L's such that $L \in [V]^4$ and $|L \cap M| \ge 3$ is at most $\binom{m}{3}(n-3) \le \frac{m^3 n}{6}$. Again the number of $\hat{M}$'s such that $\hat{M} \in [V]^m$ and $|M \cap \hat{M}| \ge 3$ is trivially less than $\binom{n}{m} \le \left(\frac{ne}{m}\right)^m$. Thus,

$$\begin{aligned}\sum_{|L\cap M|\ge 3} y_L \Pr(A_L) \quad &+ \sum_{|\hat{M}\cap M|\ge 3} z_{\hat{M}} \Pr(B_{\hat{M}}) \\ &\le \frac{m^3 n}{6}\left(1+\frac{1}{n}\right)\frac{6c^2}{n^2} + \left(\frac{ne}{m}\right)^m \exp\left(\frac{c}{100}\frac{m^3}{n}\right)\exp\left(-\frac{c}{50}\frac{m^3}{n}\right) \\ &= \left(1+\frac{1}{n}\right)\frac{c^2 m^3}{n} + \exp\left(m\Big(-\frac{c}{100}\frac{m^2}{n} + \ln\left(\frac{ne}{m}\right)\Big)\right) \\ &\le \left(1+\frac{1}{n}\right)\frac{c^2 m^3}{n} + \exp(-\sqrt{n}). \qquad (13)\end{aligned}$$

Since $c < \frac{99}{10000}$ (recall $c = \frac{1}{102}$), then for n large enough $\big(1+\frac{1}{n}\big)c < \frac{99}{10000}$ as well. Consequently,

$$\sum_{|L\cap M|\geq 3} y_L \Pr(A_L) \quad + \sum_{|\hat{M}\cap M|\geq 3} z_{\hat{M}} \Pr(B_{\hat{M}}) \leq (13) \leq \frac{99}{100}\frac{c}{100}\frac{m^3}{n} = \frac{99}{100}\ln(z_M).$$

This completes the proof of Proposition 3.3.

□

Proof of Theorem 1.3 (lower bound). We show that for sufficiently large n the following inequality holds:

$$(1-o(1))\log_2\log_2 n \leq \omega(n,3,2). \tag{14}$$

Let c, ε (with $\frac{1}{2} > \varepsilon > 0$), and n_0 be from Proposition 3.3. Let c_1 be a positive constant such that $\frac{c}{2n}\binom{m}{3} \geq c_1\frac{m^3}{n}$, for $n \geq n_0$ and $n \geq m \geq n^{\frac{1}{2}+\varepsilon}$. Proposition 3.3 guarantees the existence of a PSTS $\mathcal{G} = (V,\mathcal{E})$ with $|V| = n$, which satisfies

$$|\mathcal{E}\cap [M]^3| > \frac{c}{2n}\binom{m}{3} \geq c_1\frac{m^3}{n}, \tag{15}$$

whenever $M \subseteq V$ and $|M| = m \geq n^{\frac{1}{2}+\varepsilon}$ (since $c = \frac{1}{102}$ works in Proposition 3.3, $c_1 = \frac{1}{1250}$ satisfies (15)). Let $\mathsf{S} : \mathcal{E} \to [V]^2$ be a selector on $\mathcal{G}$. Then, for any $M \subseteq V$ with $m \geq n^{\frac{1}{2}+\varepsilon}$ the number of edges $\mathsf{S}(\mathcal{E})$ induced on the set M is at least $|\mathcal{E}\cap[M]^3|$. Hence,

$$|\mathsf{S}(\mathcal{E})\cap [M]^2| \geq |\mathcal{E}\cap[M]^3| \geq c_1\frac{m^3}{n}. \tag{16}$$

We construct a clique of size $\log_2\log_2 n - O(1)$. Set $M_1 = V$. Since $|M_1| = n \geq n^{\frac{1}{2}+\varepsilon}$, (16) yields that

$$|\mathsf{S}(\mathcal{E})\cap [M_1]^2| \geq c_1\frac{n^3}{n} = c_1 n^2.$$

Consequently, there must be an element $a_1 \in M_1$ and a set $M_2 \subseteq M_1$ with $|M_2| \geq \frac{2c_1n^2}{n} = 2c_1 n$ such that $\{a_1, x\} \in \mathsf{S}(\mathcal{E})$ for any $x \in M_2$.

Set $c_2 = 2c_1$. Then, $|M_2| \geq c_2 n$. If $c_2 n \geq n^{\frac{1}{2}+\varepsilon}$, then (16) infers that

$$|\mathsf{S}(\mathcal{E})\cap [M_2]^2| \geq c_1\frac{(c_2n)^3}{n} = c_1c_2^3n^2.$$

Thus, there must be an element $a_2 \in M_2$ and a set $M_3 \subseteq M_2$ with $|M_3| \geq \frac{2c_1c_2^3n^2}{c_2n} = 2c_1c_2^2n$ such that $\{a_2, x\} \in \mathsf{S}(\mathcal{E})$ for any $x \in M_3$.

In general, set $c_{i+1} = 2c_1c_i^2$, which leads to $c_{i+1} = (2c_1)^{2^i-1} = \big(\frac{1}{625}\big)^{2^i-1}$. We can carry on with this construction as long as $c_i n > n^{\frac{1}{2}+\varepsilon}$. If i_0 is the largest such i, then $c_{i_0}n = \Theta(n^{\frac{1}{2}+\varepsilon})$ or equivalently $625^{(2^{i_0}-1)} = \Theta(n^{\frac{1}{2}-\varepsilon})$, which yields $i_0 \geq \log_2\log_2 n - O(1)$.

□

4 Concluding remarks

Our main tool to find the lower bound on $\omega(n,3,2)$ was Proposition 3.3. In particular, for a given PSTS $\mathcal{G} = (V,\mathcal{E})$, a selector S and a set $M \subseteq V$, $|M| > n^{\frac{1}{2}+\varepsilon}$, we concluded in (16) that the number of edges $\mathsf{S}(\mathcal{E})$ induced on the set M is at least $|\mathcal{E} \cap [M]^3|$. However, it looks very likely that in general this number, *i.e.*, $|\mathsf{S}(\mathcal{E}) \cap [M]^2|$, is much bigger. In fact, there are many edges in $\mathsf{S}(\mathcal{E}) \cap [M]^2$, which are contained in triples that do not lie entirely in M. We conjecture that the right magnitude of $\omega(n, 3, 2)$ is $\log_2(n)$.

Conjecture 4.1. *There exists a constant c such that*

$$c \log_2(n) \leq \omega(n, 3, 2).$$

We believe that our proof of Theorem 1.3 can be modified to give similar bounds on $\omega(n, k, 2)$. The problem of estimating $\omega(n, k, l)$, $l \geq 3$, seems to be however harder.

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