McMaster University

Advanced Optimization Laboratory

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More bounds on the diameters
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Authors:

David Bremner, Antoine Deza,
William Hua, Lars Schewe

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Abstract

Let $\Delta(d,n)$ be the maximum possible edge diameter over all d-dimensional polytopes defined by n inequalities. The Hirsch conjecture, formulated in 1957, suggests that $\Delta(d,n)$ is no greater than $n-d$. No polynomial bound is currently known for $\Delta(d,n)$, the best one being quasi-polynomial due to Kalai and Kleitman in 1992. Goodey showed in 1972 that $\Delta(4,10)=5$ and $\Delta(5,11)=6$, and more recently, Bremner and Schewe showed $\Delta(4,11)=\Delta(6,12)=6$. In this follow-up, we show that $\Delta(4,12)=7$ and present strong evidence that $\Delta(5,12)=\Delta(6,13)=7$.

Finding a good bound on the maximal edge diameter $\Delta(d,n)$ of a polytope in terms of its dimension d and the number of its facets n is one of the basic open questions in polytope theory [7]. Although some bounds are known, the behaviour of the function $\Delta(d,n)$ is largely unknown. The Hirsch conjecture, formulated in 1957 and reported in [4], states that $\Delta(d,n)$ is linear in n and d: $\Delta(d,n)\leq n-d$. The conjecture is known to hold in small dimensions, i.e., for $d\leq 3$ [11], along with other specific pairs of d and n (Table 1). However, the asymptotic behaviour of $\Delta(d,n)$ is not well understood: the best upper bound — due to Kalai and Kleitman — is quasi-polynomial [9].

In this article we will show that $\Delta(4,12)=7$ and present strong evidence for $\Delta(5,12)=\Delta(6,13)=7$. The first of these new values is of particular interest since it indicates that the Hirsch bound is not sharp in dimension 4.

Our approach is computational and builds on the approach used by Bremner and Schewe [3]. Section 1 introduces our computational framework and some related background. We then discuss our results in Section 2.

		$n-2d$				
		0	1	2	3	4
	4	4	5	5	6	7+
	5	5	6	7-8	7+	8+
d	6	6	7-9	8+	9+	9+
	7	7-10	8+	9+	10+	11+
	8	8+	9+	10+	11+	12+

Table 1: Previously known bounds on $\Delta(d, n)$ [3, 6, 8, 12].

1 General approach

In this section we give a summary of our general approach. This is substantially similar to that in [3], and the reader is referred there for more details.

It is easy to see via a perturbation argument that $\Delta(d, n)$ is always achieved by some simple polytope. By a reduction applied from [12], we only need to consider *end-disjoint* facet-paths: paths where the end vertices do not lie on a common facet (*facet-disjointness*). It will be convenient both from an expository and a computational view to work in a polar setting where we consider the lengths of facet-paths on the boundary of simplicial polytopes. We apply the term *end-disjoint* equally to the corresponding facet paths, where it has the simple interpretation that two end facets do not intersect.

For any set $Z = \{ x_1 \dots x_{r-2}, y_1 \dots y_4 \} \subset \mathbb{R}^r$, as a special case of the Grassmann-Plücker relations [1, §3.5] on determinants we have

$$\begin{aligned} &\det(x_1 \dots x_{d-1}, y_1, y_2) \cdot \det(x_1 \dots x_{d-1}, y_3, y_4) \\ &+ \det(x_1 \dots x_{d-1}, y_1, y_4) \cdot \det(x_1 \dots x_{d-1}, y_2, y_3) \\ &- \det(x_1 \dots x_{d-1}, y_1, y_3) \cdot \det(x_1 \dots x_{d-1}, y_2, y_4) = 0 \end{aligned} \tag{1}$$

We are in particular interested in the case where $r = d + 1$ and Z represents $(d + 3)$-points in $\mathbb{R}^d$ in homogeneous coordinates; the various determinants are then signed volumes of simplices. In the case of points drawn from the vertices of a simplicial polytope, we may assume without loss of generality that these simplices are never flat (i.e. determinant 0).

Thus if we define $\chi(v_1 \dots v_{d+1}) = \text{sign}(\det(v_1 \dots v_{d+1}))$ it follows from (1) that

$$\begin{aligned}\{&\chi(x_1 \dots x_{d-1}, y_1, y_2)\chi(x_1 \dots x_{d-1}, y_3, y_4), \\ -&\chi(x_1 \dots x_{d-1}, y_1, y_3)\chi(x_1 \dots x_{d-1}, y_2, y_4), \\ &\chi(x_1 \dots x_{d-1}, y_1, y_4)\chi(x_1 \dots x_{d-1}, y_2, y_3)\} \quad = \{-1, +1\}.\end{aligned}$$

Any alternating map $\chi : E^{d+1} \to \{-, +\}$ satisfying these constraints for all $(d+3)$-subsets is called a *uniform chirotope*; this is one of the many axiomatizations of *uniform oriented matroids* [1]. The facets and interior points of a uniform chirotope are straightforward to define in terms of equality and non-equality of related signs. In the rest of this paper we call uniform chirotopes simply chirotopes.

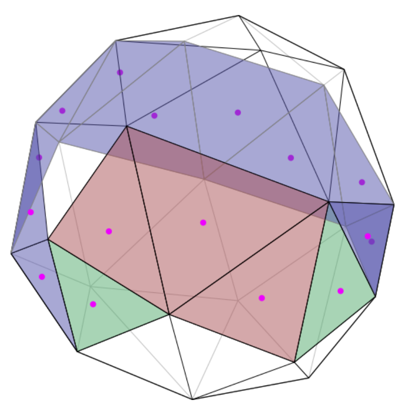

Figure 1: Illustrating a non-shortest facet-path.

Our general strategy is to show $\Delta(d, n) \neq k$ by generating all combinatorial types of facet-paths of length k on n vertices in dimension d and showing that none can be embedded on the boundary of a chirotope as a shortest path. Note that if the facet-path uses all available vertices, then there cannot be any interior points. In the general case, we solve a further relaxation of the problem, and show that even if some points are allowed to be interior, a particular combinatorial type of path is not embeddable on the boundary of the convex hull of n-points in $\mathbb{R}^d$. In addition to Grassman-Plücker constraints, and those that force the k-path onto the boundary, we also add constraints preventing the existence of shorter paths between the starting and ending facets, i.e. every potential shortcut is infeasible by virtue of containing a non-facet. See Figure 1 for an illustration of a shortcut on a 3-dimensional polytope.

Chirotopes can be viewed as a generalization of real polytopes in the sense that for every real polytope, we can obtain its chirotope directly. Therefore, showing the non-existence of chirotopes satisfying specific properties immediately precludes the existence of real polytopes holding the same properties. The search for a chirotope with a particular facet-path on its boundary is encoded as an instance of SAT [13, 14]. The SAT solver used here was MiniSat [5].

The generation of all possible paths for particular d and n begins with case where the paths are *non-revisiting*, i.e., paths where no vertex is visited more than once. These can be generated via a simple recursive scheme, using a bijection with *restricted growth strings*.

Multiple revisit paths are generated from paths with one less revisit by identifying pairs of vertices without introducing extra ridges to the facet-path or causing the end facets to intersect.

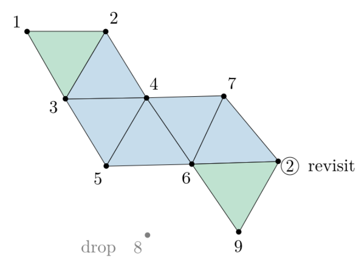

Figure 2: Example of a facet-path.

If a vertex is not used in a facet-path we call this occurrence a *drop*. See Figure 1 for an illustration of a path of length 6 involving 1 revisit (vertex 2) and and 1 drop (vertex 8) with $n = 9$ and $d = 3$. We can then classify paths by dimension d, primal-facets/dual-vertices n, length k, the number of revisits m, and the number of drops l. For end-disjoint paths, a simple counting argument yields:

$$\begin{aligned} m - l &= k + d - n \\ m &\leq k - d \\ l &\leq n - 2d \end{aligned}$$

Table 2 provides the number of paths to consider for each possible combination of values.

With the implementation of [3], we were able to reconfirm Goodey's results for $\Delta(4, 10)$ and $\Delta(5, 11)$ in a matter of minutes. While the number of paths to consider increases with the number of the revisits, in our experiments these paths are much less computationally

d	n	k	m	l	#
4	10	6	0	0	15
4	10	6	1	1	24
4	10	6	2	2	16
4	11	7	0	0	50
4	11	7	1	1	200
4	11	7	2	2	354
4	11	7	3	3	96
4	12	8	0	0	160
4	12	8	1	1	1258
4	12	8	2	2	5172

d	n	k	m	l	#
4	12	8	3	3	7398
4	12	8	4	4	1512
5	11	7	1	0	98
5	11	7	2	1	98
5	12	8	1	0	1079
5	12	8	2	1	3184
5	12	8	3	2	2904
6	12	7	1	0	11
6	13	8	1	0	293
6	13	8	2	1	452

Table 2: Number of paths to consider, SAT instances to solve.

demanding than the ones with fewer revisits. For example, the 7,398 paths of length 8 on 4-polytopes with 12 facets and involving 3 revisits and 3 drops require only a tiny fraction of the computational effort to tackle the 160 paths without a drop or revisit.

In order to deal with the intractability of the problem as the dimension, number of facets, and path length increased, we proceeded by splitting our original facet embedding problem into subproblems by fixing chirotope signs. We use the (non-SAT based) `mpc` backtracking software [2] to backtrack to a certain fixed level of the search tree; every leaf job was then processed in parallel on the Shared Hierarchical Academic Research Computing Network (SHARCNET). Figure 3 illustrates the splitting process on a problem generated from the octahedron. Note that variable propagation reduces the number of leaves of the tree.

Jobs requiring a long time to complete were further split and executed on the cluster until the entire search space was covered. Table 3 provides the number of paths which were computationally difficult enough to require splitting. For example, out of 160 paths of length 8 on 4-polytopes with 12 facets without drop or revisit, 2 required splitting.

2 Results

Summarizing the computational results, we have:

Proposition 1. *There are no* $(4, 12)$*- polytopes with facet-disjoint vertices at distance 8.*

Note that we actually prove something slightly stronger: no chirotope admits a path

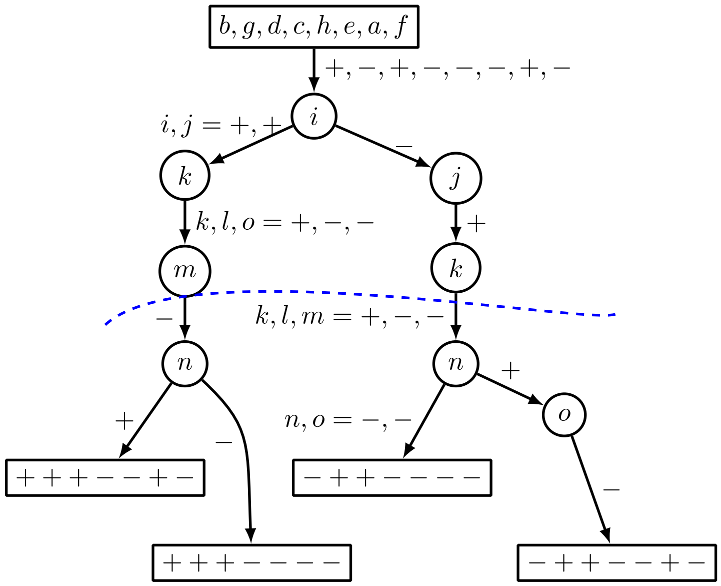

Figure 3: Using partial backtracking to generate subproblems

d	n	k	m	l	#
4	12	8	0	0	2
5	12	8	1	0	15
5	12	8	2	1	6
6	13	8	1	0	138
6	13	8	2	1	63

Table 3: Number of "difficult" paths.

of length 8 between vertex-disjoint facets on its boundary for $d = 4, n = 12$, i.e., there are no so-called $(4, 12)$-*matroid polytopes* with vertex-disjoint facets at distance 8. While the non-existence of k-length paths implies the non-existence of $(k+1)$-length paths, it is not obvious if the non-existence of end-disjoint k-length paths implies the non-existence of $(k+1)$-length paths. To be able to rule out vertices (not necessarily facet-disjoint) at distance $l > k$, we introduce the following lemma.

Lemma 1. *If $\Delta(d-1, n-1) < k$ and there is no (d, n)-polytope with two facet-disjoint vertices at distance k, then $\Delta(d, n) < k$.*

Proof. Assume the contrary. Let u and v be vertices on a (d, n)-polytope at distance $l \geq k$. By considering a shortest path from u to v, there is a vertex w at distance k from u. u and

w must share a common facet F to prevent a contradiction. F is a $(d-1, n-1)$-polytope with diameter at least k. □

By Proposition 1 and because $\Delta(3, 11) = 6$ [11], we can apply Lemma 1 to obtain the following new entry for $\Delta(d, n)$.

Corollary 1. $\Delta(4, 12) = 7$

The computations for $\Delta(5, 12)$ and $\Delta(6, 13)$ are still underway. In particular, out of the 7,167 8-paths to consider for 5-polytopes having 12 facets, only 11 paths with 1 revisit and no drop remain to be computed. If the results for remaining 8-paths keep on showing unsatisfiability, it would imply that $\Delta(5, 12) \neq 8$ and $\Delta(6, 13) \neq 8$. Since $7 \leq \Delta(5, 12) \leq 8$ [3, 8], by Proposition 1 we could immediately obtain $\Delta(5, 12) = 7$. We recall the following result of Klee and Walkup [12]:

Property 1. $\Delta(d, 2d + k) \leq \Delta(d - 1, 2d + k - 1) + \lfloor k/2 \rfloor + 1$ *for* $0 \leq k \leq 3$

Applying Property 1 to $\Delta(5, 12) = 7$ would yield a new upper bound $\Delta(6, 13) \leq 8$, from which we could obtain $\Delta(6, 13) = 7$. Property 1 along with the 3 new entries for $\Delta(d, n)$ would imply the additional upper bounds: $\Delta(5, 13) \leq 9$, $\Delta(6, 14) \leq 11$, $\Delta(7, 14) \leq 8$, $\Delta(7, 15) \leq 12$ and $\Delta(8, 16) \leq 13$ (see Table 4).

		$n - 2d$				
		0	1	2	3	4
	4	4	5	5	6	**7**
	5	5	6	**7**	**7-9**	8+
d	6	6	**7**	**8-11**	9+	9+
	7	**7-8**	**8-12**	9+	10+	11+
	8	**8-13**	9+	10+	11+	12+

Table 4: Summary of bounds on $\Delta(d, n)$ assuming $\Delta(5, 12) = \Delta(6, 11) = 7$.

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References

[1] Anders Björner, Michel Las Vergnas, Bernd Sturmfels, Neil White, and Gunter M. Ziegler, editors. *Oriented Matroids.* Cambridge University Press, second edition, 1999.

[2] David Bremner, Jürgen Bokowski, and Gábor Gévay. *Symmetric matroid polytopes and their generation*, European Journal of Combinatorics, **30** (2009) no. 8, pp 1758–1777.

[3] David Bremner and Lars Schewe, *Edge-graph diameter bounds for convex polytopes with few facets*, submitted, available at `arXiv:0809.0915`.

[4] George B. Dantzig, *Linear programming and extensions.* Princeton University Press, Princeton, N.J. (1963).

[5] Niklas Eén and Niklas Sörensson, MiniSat HP, `http://minisat.se/`

[6] Paul R. Goodey, *Some upper bounds for the diameters of convex polytopes*, Israel Journal of Mathematics **11** (1972), no. 4, 380-385.

[7] Branko Grünbaum, *Convex polytopes*, 2nd ed., Graduate Texts in Mathematics, vol. 221, Springer-Verlag, New York, 2003. Prepared and with a preface by Volker Kaibel, Victor Klee and Günter M. Ziegler, pp. 341-355b.

[8] Fred Holt and Victor Klee, *Many polytopes meeting the conjectured Hirsch bound*, Discrete and Computational Geometry **20** (1998), 1-17.

[9] Gil Kalai and Daniel J. Kleitman, *A quasi-polynomial bound for the diameter of graphs of polyhedra*, Bulletin of the American Mathematical Society **26** (1992), no. 2, 315-316.

[10] Edward D. Kim and Francisco Santos, *An update on the Hirsch conjecture: Fifty-two years later*, available at `arXiv:0907.1186`.

[11] Victor Klee, *Diameters of polyhedral graphs*, Canadian Journal of Mathematics **16** (1964), 602-614.

[12] Victor Klee and David W. Walkup, *The d-step conjecture for polyhedra of dimension* $d < 6$, Acta Mathematica **117** (1967), no. 1, 53-78.

[13] Lars Schewe, *Satisfiability Problems in Discrete Geometry*, Dissertation, TU Darmstadt, 2007.

[14] Lars Schewe, *Non-realizable minimal vertex triangulations of surfaces: Showing non-realizability using oriented matroids and satisfiability solvers* Discrete and Computational Geometry (2009). `http://dx.doi.org/10.1007/s00454-009-9222-y`, In Press.

David Bremner

Faculty of Computer Science, University of New Brunswick.

Box 4400, Fredericton NB, Canada

Email: bremner@unb.ca

Antoine Deza, William Hua

Advanced Optimization Laboratory, Department of Computing and Software, McMaster University, Hamilton, Ontario, Canada.

Email: deza, huaw@mcmaster.ca

Lars Schewe

Fachbereich Mathematik, Technische Universität Darmstadt, Germany.

Email: schewe@mathematik.tu-darmstadt.de