Computational and Geometric Aspects of Linear Optimization

Computational and Geometric Aspects of Linear Optimization

By

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A Thesis

Submitted to the School of Graduate Studies

in Partial Fulfilment of the Requirements

for the Degree

Doctorate of Philosophy

McMaster University

DOCTORATE OF PHILOSOPHY (2011) McMaster University
(Computing and Software) Hamilton, Ontario

TITLE:	Computational and Geometric Aspects of Linear Optimization
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SUPERVISOR:	Dr. Antoine Deza
NUMBER OF PAGES:	xii, 67.

Abstract

This thesis deals with combinatorial and geometric aspects of linear optimization, and consists of two parts.

In the first part, we address a conjecture formulated in 2008 and stating that the largest possible average diameter of a bounded cell of a simple hyperplane arrangement of n hyperplanes in dimension d is not greater than the dimension d. The average diameter is the sum of the diameters of each bounded cell divided by the total number of bounded cells, and then we consider the largest possible average diameter over all simple hyperplane arrangements. This quantity can be considered as an indication of the average complexity of simplex methods for linear optimization. Previous results in dimensions 2 and 3 suggested that a specific type of extensions, namely the covering extensions, of the cyclic arrangement might achieve the largest average diameter. We introduce a method for enumerating the covering extensions of an arrangement, and show that covering extensions of the cyclic arrangement are not always among the ones achieving the largest diameter.

The software tool we have developed for oriented matroids computation is used to exhibit a counterexample to the hypothesized minimum number of external facets of a simple arrangement of n hyperplanes in dimension d; i.e. facets belonging to exactly one bounded cell of a simple arrangement. We determine the largest possible average diameter, and verify the conjectured upper bound, in dimensions 3 and 4 for arrangements defined by no more than 8 hyperplanes via the associated uniform oriented matroids formulation. In addition, these new results substantiate the hypothesis that the largest average diameter is achieved by an arrangement minimizing the number of external facets.

The second part focuses on the colourful simplicial depth, i.e. the number of colourful simplices in a colourful point configuration. This question is closely related to the colourful linear programming problem. We show that any point in the convex hull of each of $(d+1)$ sets of $(d+1)$ points in general position in $\mathbb{R}^d$ is contained in at least $\lceil (d+1)^2/2 \rceil$ simplices with one vertex from each set. This improves the previously established lower bounds for $d \geq 4$ due to Bárány in 1982, Deza et al in 2006, Bárány and Matoušek in 2007, and

Stephen and Thomas in 2008.

We also introduce the notion of octahedral system as a combinatorial generalization of the set of colourful simplices. Configurations of low colourful simplicial depth correspond to systems with small cardinalities. This construction is used to find lower bounds computationally for the minimum colourful simplicial depth of a configuration, and, for a relaxed version of the colourful depth, to provide a simple proof of minimality.

Acknowledgments

The thesis was written under the guidance and with the help of my supervisor, Dr. Antoine Deza, whose valuable advices and extensive knowledge helped me all along. I am indebted to him for his continuous encouragement and generous support in many aspects of my life. My special thanks go to the members of the supervisory and defence committees: Antoine Deza, Frantisek Franek, Bartosz Protas, Tamás Terlaky, and Hugh Thomas.

I sincerely thank Tamon Stephen, David Bremner, William Hua, David Forge, Hiroki Nakayama, Komei Fukuda, Sonoko Moriyama, Hiroyuki Miyata and Grant Custard for many insightful discussions and useful suggestions.

I appreciate the great aid and warm support from all the members of the Advanced Optimization Laboratory. They made my life at McMaster much easier and more memorable.

Furthermore, I thank the Natural Sciences and Engineering Research Council of Canada (NSERC) for the financial support through its Postgraduate Scholarship (PGS) program.

Finally, I am deeply indebted to my family and their patience, understanding and continuous support.

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Notations

$\delta(P)$	diameter of a polytope P
$\delta(\mathcal{A})$	average diameter of a bounded cell of $\mathcal{A}$
$\Delta_{\mathcal{A}}(d,n)$	largest possible average diameter of a bounded cell of $\mathcal{A}_{d,n}$
Ω	colourful octahedron
$\phi(\mathcal{A})$	number of external facets of $\mathcal{A}$
$\Phi_{\mathcal{A}}(d,n)$	smallest possible number of external facets of $\mathcal{A}_{d,n}$
$X \circ Y$	composition of circuits X and Y
d-object	object of dimension d
$\mathcal{A}_{d,n}$	simple arrangement formed by n hyperplanes in $\mathbb{R}^d$
$\text{conv}(S)$	convex hull of a set of points S
$Dep(V)$	space of linear dependencies of vector configuration V
$\text{depth}(p)$	colourful simplicial depth of point p
det(A)	determinant of the matrix A
$f_k(P)$	number of k-faces of polytope P
$f_k(\mathcal{A})$	number of k-faces of arrangement $\mathcal{A}$
$f_k^+(\mathcal{A})$	number of k-faces that are inside the envelope of arrangement $\mathcal{A}$
$f_k^0(\mathcal{A})$	number of k-faces that are on the envelope of arrangement $\mathcal{A}$
$\text{int}(S)$	interior of the set S
$r(A)$	rank of elements A in a matroid or in an oriented matroid
$r^*(A)$	rank of elements A in the dual of a matroid or of an oriented matroid
$\mathbb{R}^d$	d dimensional Euclidean space
sign(a)	sign of the real number a
$\mu(d)$	minimum colourful simplicial depth in dimension d
$\mu^{\Diamond}(d)$	minimum colourful simplicial depth in dimension d with a generalized core
$\mathbb{S}^{d-1}$	d-sphere

Chapter 1

Preliminaries

1.1 Polytopes

A *hyperplane* is defined by $\{\mathbf{x} \in \mathbb{R}^d \mid \mathbf{a}^T\mathbf{x} = c\}$ for some $\mathbf{a} \in \mathbb{R}^d$ ($\mathbf{a} \neq \mathbf{0}$) and $c \in \mathbb{R}$. If the equality is replaced with inequality, we have a *halfspace* .

Definition 1.1.1 *A polyhedron is an intersection of finitely many closed half-spaces. A polytope is a bounded polyhedron.*

Let P be a d-polyhedron, i.e., a polyhedron of dimension d. A *face* of P is the intersection of P with some *supporting hyperplane*, a hyperplane that intersects P and has all the points of P contained in one of the two closed halfspaces determined by the hyperplane. The 0-faces, 1-faces, and $(d-1)$-faces are called *vertices*, *edges* and *facets* respectively. The number of k-faces of P is denoted by $f_k(P)$ for $k = 0, \dots, d-1$ and, considering the improper face P and the empty set, we have $f_d(P) = f_{-1}(P) = 1$.

Given a d-polytope P defined by n inequalities, its *diameter*, denoted by $\delta(P)$, is the minimum number of edges needed to connect any pair of vertices.

1.2 Arrangements

Definition 1.2.1 *An arrangement $\mathcal{A}_{d,n}$ in $\mathbb{R}^d$ is a family of $n \geq d+1$ hyperplanes.*

A *linear arrangement* consists of hyperplanes containing the origin. An arrangement is *simple* if any d hyperplanes intersect at a distinct point.

Remark 1.2.2 *In a simple arrangement in $\mathbb{R}^d$, no two hyperplanes are parallel to each other and no $d+1$ hyperplanes intersect at one point.*

In this thesis we consider simple arrangements. The d-polyhedra defined by the hyperplanes of an arrangement $\mathcal{A}_{d,n}$ are called the *cells* of $\mathcal{A}_{d,n}$. The k-faces of $\mathcal{A}_{d,n}$ are the k-faces of its cells. Let $f_k(\mathcal{A}_{d,n})$ denote the number of k-faces for $k = 0, \ldots, d-1$ and $f_d(\mathcal{A}_{d,n})$ the number of cells of $\mathcal{A}_{d,n}$.

The bounded facets of an unbounded cell are called *external*, and the facets shared by two bounded cells are called *internal*. The set of external facets is referred to as the *envelope* of the arrangement. The faces of the external facets are called *external*, and the non-external bounded faces are called *internal*. The internal cells are the bounded ones. Let $f_k^0(\mathcal{A}_{d,n})$ denote the number of external k-faces, and $f_k^+(\mathcal{A}_{d,n})$ the number of the internal k-faces. The number of k-faces and bounded cells are known for a simple arrangement.

Theorem 1.2.3 *For $k = 0, 1, \ldots, d$, a simple arrangement $\mathcal{A}_{d,n}$ has $f_k(\mathcal{A}_{d,n}) = \sum_{i=0}^{k} \binom{d-i}{k-i}\binom{n}{d-i}$ k-faces.*

Theorem 1.2.4 *A simple arrangement $\mathcal{A}_{d,n}$ has $f_d^+(\mathcal{A}_{d,n}) = \binom{n-1}{d}$ bounded cells, and $f_{d-1}^+(\mathcal{A}_{d,n}) + f_{d-1}^0(\mathcal{A}_{d,n}) = n\binom{n-2}{d-1}$ bounded facets.*

General references for polytopes and hyperplane arrangements are the books of Edelsbrunner [24], Grunbaum [38], and Ziegler [63].

1.3 Oriented matroids

Oriented matroids form an abstraction and generalization of the combinatorial structure of hyperplane arrangements, as well as some other discrete geometrical objects such as vector and point configurations. We first give a brief introduction to matroids [52] on which oriented matroids are based.

1.3.1 Matroids

We show two equivalent definitions of matroids.

Definition 1.3.1 (independent sets) *A matroid $\mathcal{M}$ is an ordered pair $(E, \mathcal{I})$ where E is a finite set, and $\mathcal{I}$ is a collection of subsets of E satisfying the following three conditions:*

$(I1)$ $\phi \in \mathcal{I}$.

$(I2)$ *If $X \in \mathcal{I}$ and $Y \subset X$, then $Y \in \mathcal{I}$.*

$(I3)$ *If $X, Y \in \mathcal{I}$ and $|Y| < |X|$, then there exists an element $e \in X \setminus Y$ such that $Y \cup \{e\} \in \mathcal{I}$.*

The set E is called the *ground set*. The collection $\mathcal{I}$ is called the *independent sets* and the three conditions are referred to as the *independent sets axioms*. The Axiom $(I3)$ is also known as the *independence augmentation axiom*. A maximal independent set is called a *basis*, and the independence augmentation axiom indicates that all bases have the same cardinality, which is the *rank* of the matroid, denoted by $r(E)$. Similarly we can define the rank of a subset A of the ground set E, denoted by $r(A)$. Two sets $A, B \subseteq E$ are *comodular* if $r(A) + r(B) = r(A \cup B) + r(A \cap B)$.

Example 1 *Figure 1.1 shows a pair of non-comodular hyperplanes H^1 and H^2. $H^1 = 12$ and $H^2 = 34$, both with rank 2, while $H^1 \cap H^2 = \phi$ and $H^1 \cup H^2 = 1234$, with ranks 0 and 3 respectively.*

A matroid is said to be *uniform* if any set of $r(E)$ elements is a basis. A *flat* is a set $F \subseteq E$ satisfying $\forall\ e \in E : r(F \cup e) = r(F) \Rightarrow e \in F$, i.e., all the elements residing in a certain "subspace". A *hyperplane* is a maximal proper flat, whose rank is $r(E) - 1$.

Given a matroid $\mathcal{M}$, its dual, denoted by $\mathcal{M}^*$, has the same ground set as $\mathcal{M}$ with the bases being the complements of the bases of $\mathcal{M}$. We use $r^*(A)$, where $A \subseteq E$, to denote the rank of A in the dual matroid.

A matroid can also be defined in terms of minimal dependent sets, or *circuits*.

Definition 1.3.2 (circuits) *A matroid $\mathcal{M}$ is an ordered pair $(E, \mathcal{C})$ where E is a finite set and $\mathcal{C}$ is a collection of subsets of E satisfying the following three conditions:*

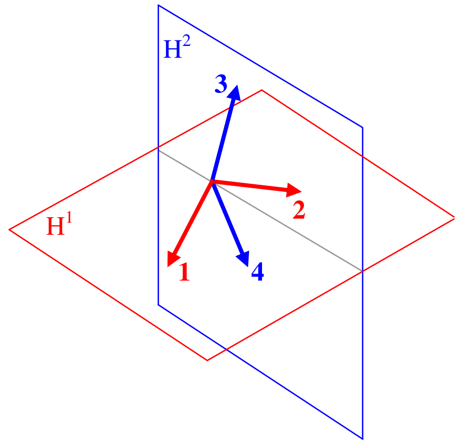

Figure 1.1: A non-comodular pair

(C1) $\phi \notin \mathcal{C}$.

(C2) *If* $X \in \mathcal{C}$ *and* $Y \subset X$, *then* $Y \notin \mathcal{C}$.

(C3) *If* $X, Y \in \mathcal{C}$ *and* $e \in X \cap Y$, *then there exists a* $Z \in \mathcal{C}$ *such that* $Z \subseteq \{X \cup Y\} - \{e\}$.

The conditions above are referred to as *circuit axioms*. Given a matroid $\mathcal{M}$ and its dual $\mathcal{M}^*$, the circuits $\mathcal{C}^*$ of $\mathcal{M}^*$ are called the *cocircuits* of $\mathcal{M}$.

1.3.2 Oriented matroids

While the matroid abstracts only the linear dependency among discrete elements, the oriented matroid tells more about the dependency by assigning a sign to each element in a dependent set. Therefore, oriented matroids are more suitable for representing discrete geometric objects such as point configurations and arrangements, for which the relative positions of the elements are essential.

Many terms used for matroids, including *ground set*, *circuit*, *basis*, *rank*, *hyperplane*[1] and *duality*, can be naturally extended to oriented matroids. An oriented matroid circuit is represented by a signed set of the elements in the

[1]A hyperplane in the context of matroids or oriented matroids is a discrete object - a subset of the ground set satisfying certain conditions.

corresponding minimal dependent set. Given a circuit X, we use X^+ and X^- to denote the elements associated with the positive and the negative sign respectively. The following definition of oriented matroids derives from the circuits definition of matroids.

Definition 1.3.3 *An Oriented matroid is denoted by $\mathcal{M} = (E, \mathcal{C})$, where E is the ground set, and $\mathcal{C}$ is a set of circuits satisfying the following four conditions (circuit axioms).*

(C1) $\phi \notin \mathcal{C}$.

(C2) *If* $X \in \mathcal{C}$, *then* $-X \in \mathcal{C}$.

(C3) *If* $X \in \mathcal{C}$ *and* $Y \subset X$, *then* $Y \notin \mathcal{C}$.

(C4) *If* $X, Y \in \mathcal{C}$ *with* $X \neq Y$ *and* $e \in X^+ \cap Y^-$, *then there exists a* $Z \in \mathcal{C}$ *such that* $Z^+ \subseteq (X^+ \cup Y^+) \backslash \{e\}$ *and* $Z^- \subseteq (X^- \cup Y^-) \backslash \{e\}$.

For some element $e \in E$ and $X \in \mathcal{C}$, we use X_e to denote the sign of e in X. Two circuits X and Y are *conformal* if $X_e Y_e \geq 0$ for all $e \in E$.

The *vectors* of an oriented matroid are *composed* repeatedly from the circuits in the following way:

$$(X \circ Y)_e = \begin{cases} X_e & if\, X_e \neq 0, \\ Y_e & otherwise. \end{cases}$$

The linear dependencies of a *vector configuration* $V = \{v_1, v_2, \cdots, v_n\} \subseteq \mathbb{R}^d$ provide a classical example of an oriented matroid. Let $V \in \mathbb{R}^{d \times n}$ be the matrix of the n vectors. Then the space of linear dependencies is

$$Dep(V) := \{u \in \mathbb{R}^n \mid Vu = \mathbf{0}\}. \tag{1.3.1}$$

The vectors in the corresponding oriented matroid are the sign vectors in $Dep(V)$ and the circuits are the sign vectors of the minimal dependencies in $Dep(V)$.

Example 2 *For the following vector configuration (see Figure 1.2)*

$$V = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\},$$

the circuits are $\left\{\begin{pmatrix}+\\-\\+\\0\end{pmatrix},\begin{pmatrix}+\\-\\0\\+\end{pmatrix},\begin{pmatrix}+\\0\\-\\+\end{pmatrix},\begin{pmatrix}0\\+\\-\\+\end{pmatrix}\right\}$ *and the negations of them. The vectors (of the oriented matroid) consist of* $\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$*, the circuits and the compositions of the circuits, i.e., dependencies that are not minimal. The non-minimal dependencies are* $\left\{\begin{pmatrix}+\\+\\-\\+\end{pmatrix},\begin{pmatrix}+\\-\\+\\+\end{pmatrix},\begin{pmatrix}+\\-\\+\\-\end{pmatrix},\begin{pmatrix}+\\-\\-\\+\end{pmatrix}\right\}$ *and their negations.*

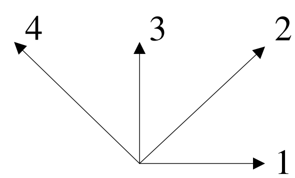

Figure 1.2: A vector configuration of 4 vectors

A linear arrangement naturally corresponds to an oriented matroid, as a linear arrangement is uniquely determined by the norm vectors of its hyperplanes, and the norm vectors form a vector configuration.

1.3.3 Pseudo-sphere arrangements

A *pseudo-sphere* arrangement is the intersection of a linear pseudo-hyperplane arrangement and the unit sphere, where a *pseudo-hyperplane* can be viewed as a perturbation of a hyperplane with certain constraints. The intersection of a linear pseudo-hyperplane and the unit sphere is a pseudo-sphere in the lower dimension. While giving orientation to the pseudo-hyperplane, one of the two *pseudo-hemispheres* is considered to be positive. Therefore, a pseudo-sphere arrangement is also called an arrangement of pseudo-hemispheres. Figure 1.3 illustrates the pseudo-sphere arrangement representation of $\mathcal{M}_{3,4}$.

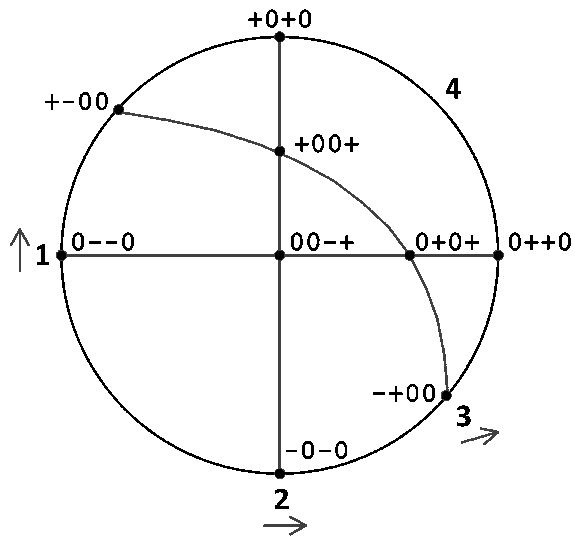

Figure 1.3: Pseudo-sphere arrangement representation of $\mathcal{M}_{3,4}$

The pseudo-sphere arrangement does not assume linearity and provide a good representation of the oriented matroid by the topological representation theorem [30]. This geometric representation allows us to interpret the k-faces of an oriented matroid. The following theorem gives an upper bound for the number of k-faces in an oriented matroid [36]:

Theorem 1.3.4 *For an oriented matroid $\mathcal{M}$ of rank $d+1$, $f_k(\mathcal{M}) \leq \binom{d}{k} f_d(\mathcal{M})$, for $k = 0, 1, \ldots, d$.*

1.3.4 Chirotopes

For a vector configuration $\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$ in $\mathbb{R}^r$, the *chirotope*, or *basis orientation*, is defined by the signs of the determinants of the ordered r-subset of the vectors, i.e., $\chi : \Lambda(n, r) \to \{+, -, 0\}$ and $\chi(i_1, \ldots, i_r) := \text{sign}\det(\mathbf{v_{i_1}}, \ldots, \mathbf{v_{i_r}}) \in \{+, -, 0\}$.

Example 3 *For a vector configuration* $\{\mathbf{v_1}, \ldots, \mathbf{v_n}\}$ *in* $\mathbb{R}^2$,

$$\begin{aligned}\chi(i,j) &= \operatorname{sign}\det[\mathbf{v_i}, \mathbf{v_j}] \\ &= \operatorname{sign}\det\begin{pmatrix} \|\mathbf{v_i}\|\cos\theta_i & \|\mathbf{v_j}\|\cos\theta_j \\ \|\mathbf{v_i}\|\sin\theta_i & \|\mathbf{v_j}\|\sin\theta_j \end{pmatrix} \\ &= \operatorname{sign}\|\mathbf{v_i}\|\,\|\mathbf{v_j}\|\sin(\theta_j - \theta_i).\end{aligned}$$

In other words, $\chi(i,j) = 0$ *if* $\mathbf{v_i}$ *and* $\mathbf{v_j}$ *are colinear;* $\chi(i,j) = +$ *if the angle distance from* $\mathbf{v_i}$ *to* $\mathbf{v_j}$ *is less than* π*, or* $\mathbf{v_i}$ *can be rotated counter-clockwise to the position of* $\mathbf{v_j}$ *by an angle less than* π*;* $\chi(i,j) = +$ *if the angle distance from* $\mathbf{v_i}$ *to* $\mathbf{v_j}$ *is bigger than* π*.*

The generalization of the chirotope leads to another representation of the oriented matroid. While the circuits definition generalizes the dependencies, the chirotope definition generalizes the basis orientation[2]. The equivalence the two definitions was established by Lawrence [43].

Definition 1.3.5 *An oriented matroid is denoted by* $\mathcal{M} = (E, \chi)$*, where* E *is the ground set, and* χ*, the chirotope, is a mapping* $\chi : E^r \to \{-1, 0, 1\}$ *satisfying the following conditions:*

(X1) χ *is not identically zero.*

(X2) $\chi(e_{\sigma(1)}, e_{\sigma(2)}, \ldots, e_{\sigma(r)}) = \operatorname{sign}(\sigma)\chi(e_1, e_2, \ldots, e_r)$ *for all* $e_1, e_2, \ldots, e_r \in E$ *and permutation* σ*.*

(X3) for all $e_1, e_2, \ldots, e_r, f_1, f_2, \ldots, f_r \in E$ *such that*
$\chi(f_i, e_2, e_3, \ldots, e_r)\chi(f_1, f_2, \ldots, f_{i-1}, e_1, f_{i+1}, f_{i+2} \ldots, f_r) \geq 0$
for $i = 1, 2, \ldots, r$*,*
we have $\chi(e_1, e_2, \ldots, e_r)\chi(f_1, f_2, \ldots, f_r) \geq 0$.

An oriented matroid is *uniform* if its chirotope has no mappings to 0. In the case of a vector configuration in $\mathbb{R}^r$, uniformity means that any r vectors in the configuration are linearly independent, i.e., there is no degeneracy.

[2]The *basis* here, any set of r elements where r is the rank of the oriented matroid, is not exactly the same as the basis of an oriented matroid. They coincide only when the oriented matroid is uniform.

1.3.5 Realizable oriented matroids

The oriented matroid is not a perfect model for the geometric objects that require linearity. Every vector configuration, or arrangement, corresponds to an oriented matroid, but not vice versa. The ones that have a vector configuration representation are called *realizable*[3]. The formal definition of oriented matroid realizability follows.

Definition 1.3.6 *A realization of oriented matroid $\mathcal{M}$ of rank r on E is a mapping $\phi : E \to \mathbb{R}^r$ such that*

$$\chi(e_1, e_2, \cdots, e_r) = \operatorname{sign}\det(\phi(e_1), \phi(e_2), \cdots, \phi(e_r)) \tag{1.3.2}$$

for all $e_1, e_2, \cdots, e_r \in E$.

The realizability problem of determining whether a given oriented matroid is realizable is known to be NP-hard [57]. One way to solve the problem is the method of *solvability sequences* [9]. Recent progresses in this area include the categorization of uniform $\mathcal{M}_{4,9}$ and $\mathcal{M}_{5,9}$ into realizable and unrealizable oriented matroids by Moriyama et al [18].

Remark 1.3.7 *An oriented matroid $\mathcal{M}$ of rank r on E is realizable for $r \leq 2$, and for $r = 3$ and $|E| \leq 8$.*

1.3.6 Single element extension

Geometrically, a *single element extension* [7] of an oriented matroid $\mathcal{M}_{r,n}$ corresponds to the addition of a pseudo-hemisphere to an arrangement of n pseudo-hemispheres.

Let $\mathcal{C}^*$ be the set of cocircuits of $\mathcal{M}_{r,n}$. A single element extension of $\mathcal{M}_{r,n}$ can be represented by a *cocircuit signature* $\sigma : \mathcal{C}^* \to \{+, -, 0\}$. The cocircuits $\mathcal{C}^*_{r,n+1}$ of the new oriented matroid $\mathcal{M}_{r,n+1}$ will consist of:

(C1) $\{(X, \sigma(X)) : X \in \mathcal{C}^*_{r,n}\}$.

(C2) $\{(X \circ Y, 0) : X, Y \in \mathcal{C}^*_{r,n}, \sigma(X) = -\sigma(Y) \neq 0$, X and Y are conformal and comodular $\}$.

[3]Oriented matroid realizability is also referred to as stretchability or coordinatizability.

1.4 Colourful simplicial depth

1.4.1 From simplicial depth to colourful simplicial depth

A point $p \in R^d$ has *simplicial depth* k relative to a set S if it is contained in k closed simplices generated by $(d+1)$ sets of S. This notion was introduced by Liu [44] as a statistical measure of how representative p is of S. More generally, we consider *colourful simplicial depth*, where the single set S is replaced by $(d+1)$ sets, or colours, $\mathbf{S}_1, \ldots, \mathbf{S}_{d+1}$, and a colourful simplex containing p is generated by taking one point from each set. Figure 1.4 shows a point configuration with a point p having a colourful simplicial depth of 5. We assume that $p \cup \bigcup_{i=1}^{i=d+1} \mathbf{S}_i$ are in general position.

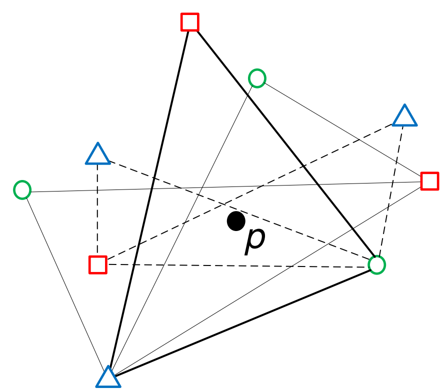

Figure 1.4: Point p with a colourful simplicial depth of 5

The colourful simplicial depth of a point p relative to $(d+1)$ point sets $\mathbf{S}_1, \ldots, \mathbf{S}_{d+1}$ gives the number of feasible solutions of the associated *colourful linear programming* problem [6, 17]: expressing p as a convex combination of points $x_1, x_2, \ldots, x_{d+1}$ with $x_i \in S_i$ for each i.

1.4.2 Normalization for colourful simplicial depth problems

As shown in [16], while exploring the colourful simplicial depth of the point p with respect to $d+1$ sets of points $\mathbf{S}_1, \ldots, \mathbf{S}_{d+1}$, we can assume the following normalizations of the problem.

(N1) Let $p = \mathbf{0}$.

(N2) $|\mathbf{S}_i| = d+1$ for each i. The sets $\mathbf{S}_1, \ldots, \mathbf{S}_{d+1}$ must each contain at least $d+1$ points for $\mathbf{0}$ to be in the interior of their convex hulls, and since we are minimizing we can assume they contain no additional points.

(N3) Scale the points of the $\mathbf{S}_i$'s so that they lie on the unit sphere $\mathbb{S}^{d-1}$. It is based on the observation that $\mathbf{0}$ is in a simplex after scaling if and only if it was in the simplex before scaling. The general position assumption can be preserved up to elementary perturbations.

Figure 1.5 illustrates the normalization of the colourful point configuration given in Figure 1.4.

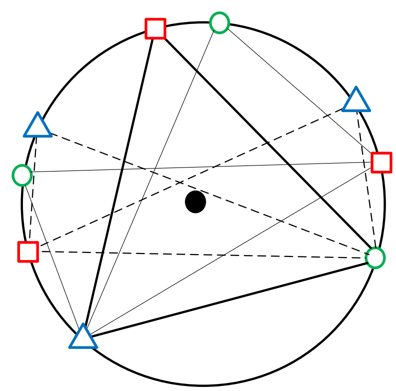

Figure 1.5: Normalized version of Figure 1.4

Part I

Hyperplane Arrangements with Large Average Diameter

Chapter 2

Introduction

2.1 Conjectured bound for the average diameter

Let $\delta(\mathcal{A})$ denote the average diameter of a bounded cell P_i of a simple arrangement $\mathcal{A}$, i.e.,

$$\delta(\mathcal{A}) = \frac{\sum_{i=1}^{f_d^+(\mathcal{A})} \delta(P_i)}{f_d^+(\mathcal{A})},$$

where $f_d^+(\mathcal{A})$ is the number of bounded cells of $\mathcal{A}$. We use $\Delta_{\mathcal{A}}(d, n)$ to denote the largest possible average diameter of a bounded cell of a simple arrangement defined by n inequalities in dimension d.

A major focus of this thesis is the following conjecture proposed by Deza, Terlaky and Zinchenko [20]:

Conjecture 2.1.1 *The average diameter of a simple arrangement is bounded by its dimension from above, i.e.,* $\Delta_{\mathcal{A}}(d, n) \leq d$.

It is shown in [22] that $\Delta_{\mathcal{A}}(2, n) = 2 - \frac{2\lceil \frac{n}{2} \rceil}{(n-1)(n-2)}$, $\Delta_{\mathcal{A}}(3, n) \geq 3 - \frac{6}{n-1} + \frac{6(\lfloor \frac{n}{2} \rfloor - 2)}{(n-1)(n-2)(n-3)}$, and d is an asymptotic lower bound for $\Delta_{\mathcal{A}}(d, n)$ for $n \to \infty$ in fixed dimension.

In order to gain further insight into the problem, we introduce a computational framework in Chapter 4 that exploits the strength of the oriented matroid abstraction to explore the combinatorial structure of an arrangement.

Previous results triggered the study of a specific type of arrangements, namely covering extensions of cyclic arrangements. A method to enumerate the covering extensions of an arrangement and related computational results are presented in Chapter 3.

2.1.1 A link to Hirsch conjecture

Hirsch Conjecture [14] states that the diameter of a d-polytope defined by n inequalities is not greater than $n-d$. While the conjecture holds for $d \leq 3$ and for $n-d \leq 6$, it was recently disproved by Santos [55] by exhibiting a counterexample for $\Delta(d,2d)$ with $d=43$ which was further improved to $d=20$ [45].

Deza, Terlaky and Zinchenko [20] noted the following link between the Hirsch conjecture and Conjecture 2.1.1. The proof of Proposition 2.1.2 shows how Hirsch conjecture relates to the average diameter of an arrangement and its envelope complexity.

Proposition 2.1.2 *If the conjecture of Hirsch holds for polytopes in dimension d, then $\Delta_{\mathcal{A}}(d,n) \leq d + \frac{2d}{n-1}$.*

Proof. Let $\{P_i \mid i = 1,2,\cdots, f_d^+(\mathcal{A}_{d,n}) = \binom{n-1}{d}\}$ denote the set of bounded cells and $f_{d-1}(P_i)$ the number of facets of P_i. Then,

$$\begin{aligned}
\delta(\mathcal{A}_{d,n}) &= \frac{1}{f_d^+(\mathcal{A}_{d,n})} \sum_{i=1}^{f_d^+(\mathcal{A}_{d,n})} \delta(P_i) \\
&\leq \frac{1}{f_d^+(\mathcal{A}_{d,n})} \sum_{i=1}^{f_d^+(\mathcal{A}_{d,n})} (f_{d-1}(P_i) - d) \qquad \text{(Hirsch conjecture)} \\
&= \frac{1}{f_d^+(\mathcal{A}_{d,n})} \sum_{i=1}^{f_d^+(\mathcal{A}_{d,n})} f_{d-1}(P_i) - d.
\end{aligned}$$

As a facet belongs to at most 2 bounded cells, $\sum_{i=1}^{f_d^+(\mathcal{A}_{d,n})} f_{d-1}(P_i)$ is at most twice the number of bounded facets in $\mathcal{A}_{d,n}$. Thus,

$$\begin{aligned}
\delta(\mathcal{A}_{d,n}) &\leq \frac{2\left(f_{d-1}^+(\mathcal{A}_{d,n}) + f_{d-1}^0(\mathcal{A}_{d,n})\right)}{f_d^+(\mathcal{A}_{d,n})} - d \\
&= \frac{2n\binom{n-2}{d-1}}{\binom{n-1}{d}} - d \qquad \text{(Theorem 1.2.4)} \\
&= d + \frac{2d}{n-1}.
\end{aligned}$$

□

2.2 Average diameter of oriented matroids

While we consider only the bounded cells of an arrangement for defining its diameter, there is no concept of boundedness for the cells of a sphere arrangement. We can define of the average diameter of oriented matroids as the average diameter of all the cells in its representative sphere arrangement. For example, the oriented matroid $\mathcal{M}_{3,4}$ shown in Figure 1.3 has average diameter $\frac{1\times4+2\times3}{7} = 1\frac{3}{7}$.

Remark 2.2.1 *The boundedness of a cell is well defined in an affine sphere arrangement, in which one of the hemispheres is chosen as the infinity hemisphere. A cell that touches the equator of an infinity hemisphere is considered to be unbounded.*

Theorem 2.2.2 *In the cases where the Hirsch conjecture holds, the average diameter of an oriented matroid $\mathcal{M}_{r,n}$ is bounded by $r-1$ from above [34].*

Proof. Given an oriented matroid $\mathcal{M}_{r,n}$, a cell in its representative sphere arrangement has dimension $r-1$. We use $f_{r-1}(\mathcal{M}_{r,n})$ to denote the number of cells of $\mathcal{M}_{r,n}$, and C_i, $i = 1, \ldots, f_{r-1}(\mathcal{M}_{r,n})$, the ith cell. We have

$$\begin{aligned}
\delta(\mathcal{M}_{r,n}) &= \frac{1}{f_{r-1}(\mathcal{M}_{r,n})} \sum_{i=1}^{f_{r-1}(\mathcal{M}_{r,n})} \delta(C_i) \\
&\leq \frac{1}{f_{r-1}(\mathcal{M}_{r,n})} \sum_{i=1}^{f_{r-1}(\mathcal{M}_{r,n})} (f_{r-2}(C_i) - (r-1)) \quad \text{(Hirsch conjecture)} \\
&= \frac{1}{f_{r-1}(\mathcal{M}_{r,n})} \sum_{i=1}^{f_{r-1}(\mathcal{M}_{r,n})} f_{r-2}(C_i) - (r-1) \\
&\leq \frac{2f_{r-2}(\mathcal{M}_{r,n})}{f_{r-1}(\mathcal{M}_{r,n})} - (r-1) \\
&\leq \frac{2(r-1)f_{r-1}(\mathcal{M}_{r,n})}{f_{r-1}(\mathcal{M}_{r,n})} - (r-1) \quad \text{(Theorem 1.3.4)} \\
&= r-1.
\end{aligned}$$

2.3 Conjectured bound for the envelope complexity

In the proof of Proposition 2.1.2, we noticed that

$$\sum_{i=1}^{f_d^+(\mathcal{A}_{d,n})} f_{d-1}(P_i) \leq 2\left(f_{d-1}^+(\mathcal{A}_{d,n}) + f_{d-1}^0(\mathcal{A}_{d,n})\right).$$

The inequality above is the result of an over-counting. Namely, as an external facet belongs to exactly one bounded facet, we have

$$\sum_{i=1}^{f_d^+(\mathcal{A}_{d,n})} f_{d-1}(P_i) = 2\left(f_{d-1}^+(\mathcal{A}_{d,n}) + f_{d-1}^0(\mathcal{A}_{d,n})\right) - f_{d-1}^0(\mathcal{A}_{d,n}). \tag{2.3.1}$$

Therefore, a lower bound for $f_{d-1}^0(\mathcal{A}_{d,n})$ would yield a tighter upper bound for $\Delta_{\mathcal{A}}(d,n)$ – assuming the Hirsch conjecture holds true for a given dimension, say $d = 4$. Let $\Phi_{\mathcal{A}}(d,n)$ be the minimum number of external facets for any simple arrangement defined by n hyperplanes in dimension d.

Conjecture 2.3.1 ([20]) *Any arrangement $\mathcal{A}_{d,n}$ has at least $d\binom{n-2}{d-1}$ external facets, i.e., $\Phi_{\mathcal{A}}(d,n) \geq d\binom{n-2}{d-1}$.*

We disprove this conjecture by providing a counter example in Chaper 4. Forge [31] established that the number of external facets of a cyclic arrangement satisfy $\phi(A^*_{d,n}) = (d+1)\binom{n-2}{d-1}$; i.e., we have $\Phi_{\mathcal{A}}(d,n) \leq (d+1)\binom{n-2}{d-1}$. It was shown that $\Phi_{\mathcal{A}}(2,n) = 2(n-1)$ for $n \geq 4$ by Deza et al [22].

Chapter 3

Covering Extension of Arrangements

We get a covering extension of an arrangement $\mathcal{A}_{d,n}$ by adding a hyperplane H such that all the vertices of $\mathcal{A}_{d,n}$ are on the same side of H. Previous results suggested that certain covering extensions of cyclic arrangements have the maximum average diameter among all $\mathcal{A}_{2,n}$ and $\mathcal{A}_{3,6}$. Figure 3.1 and 3.2 illustrates such cased in dimensions 2 and 3. This observation motivates the study of covering extensions.

3.1 Enumeration of covering extensions of arrangements

Given an arrangement $\mathcal{A}_{d,n}$, one can enumerate its covering extensions by extending its corresponding oriented matroid using the proper perturbations of the infinity element[1].

Let $\mathcal{C}^*$ be the set of cocircuits of an oriented matroid $\mathcal{M}_{r,n}$. Recall that a single element extension of $\mathcal{M}_{r,n}$ is determined by the cocircuit signature $\delta : \mathcal{C}^* \to \{+, -, 0\}$ (section 1.3.6). For ease of understanding and analysis, we use the sphere arrangement representation of oriented matroids. We can obtain the cocircuit signature δ_c corresponding to the covering extension via a series of rotations of the infinity hemisphere.

[1] We only consider simple arrangements, i.e, whose corresponding oriented matroids are uniform.

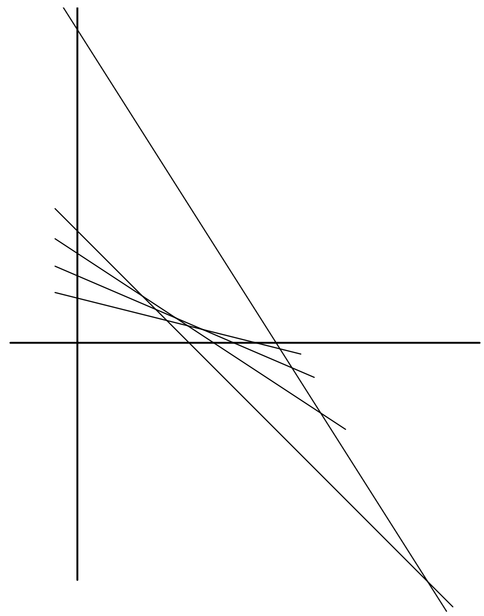

Figure 3.1: An arrangement $\mathcal{A}_{2,7}$ that maximizes the average diameter

Let H_k, $1 \leq k \leq n$, be the infinity hemisphere, and H_{n+1} the new hemisphere resulting from the perturbation of H_k. Regardless of the directions of the rotations of H_k, we keep it small enough so that the vertices which are on the positive, respectively negative, side of H_k are also on the positive, respectively negative, side of H_{n+1}, i.e., we have $\delta_c(X) = X_k$, $X \in \mathcal{C}^*$, if $X_k \neq 0$, where X_k is the kth sign of X. The signatures of the remaining cocircuits, $\delta_c(X)$ for $X \in \mathcal{C}^*$ where $X_k = 0$, are determined by how H_k is rotated.

We first choose a pair of antipodal vertices Y and $-Y$ on H_k. A vertex of a sphere arrangement is the intersection of the equators of $(r-1)$ hemispheres, and hence has $(r-1)$ zero signs. Let the ordered index set of zeros in Y, except the one at index k, be $(i_1, i_2, \ldots, i_{r-2})$. We use a sign vector $O \in \{+,-\}^{r-2}$ to represent the rotations, where O_j, $j = 1, \ldots, r-2$, records the direction of the rotation of H_k around the axis defined by the intersection of the pseudo-hyperplanes associated with H_k and H_{i_j}.

The rotations are ordered: the jth rotation is the one represented by O_j. The first rotation perturbs H_k to H_{n+1}, which moves away from all the

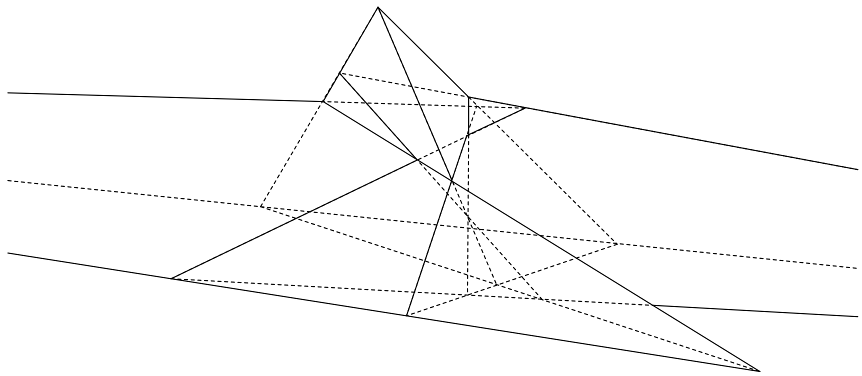

Figure 3.2: An arrangement $\mathcal{A}_{3,6}$ that maximizes the average diameter

vertices that are on the equator of H_k except the ones on the equator of H_{i_1}. The second rotation perturbs H_{n+1} away from the remaining vertices except the ones that are on the equators of H_k, H_{i_1} and H_{i_2}. At the end of the $r-2$ rotations, all vertices on H_k except Y and $-Y$ should be away from the equator of H_{n+1}. In other words, all cocircuits in $\{X \in \mathcal{C}^* : X_k = 0\}$ except Y and $-Y$ get a signature.

The signature of any cocircuit X, except for Y and $-Y$, is given by:

$$\sigma(X) = \begin{cases} X_k, & \text{if } X_k \neq 0; \\ X_{i_j}O_j, & \text{if } X_k = 0,\ j \text{ is the smallest index such that } X_{i_j} \neq 0. \end{cases}$$

Finally, for computational purposes, we set the signature of the antipodal pair Y and $-Y$ by extending the length of the orientation vector O by one and use O_{r-1} to record the chosen orientation of the antipodal pair. Thus, the signature of any cocircuit X is given by:

$$\sigma(X) = \begin{cases} O_{r-1}, & \text{if } X = Y; \\ -O_{r-1}, & \text{if } X = -Y; \\ X_k, & \text{if } X_k \neq 0; \\ X_{i_j}O_j, & \text{if } X_k = 0,\ j \text{ is the smallest index such that } X_{i_j} \neq 0. \end{cases}$$

See Figure 3.3 for an illustration of a covering extension of $\mathcal{M}_{3,4}$. The covering hemisphere, coloured in red, is obtained as a perturbation of hemisphere H_4 by choosing Y =`0++0` and $-Y$ =`0--0`, rotating the sphere around the intersection of hemispheres H_4 and H_1 towards the reader, and then perturbing the antipodal pairs clockwise.

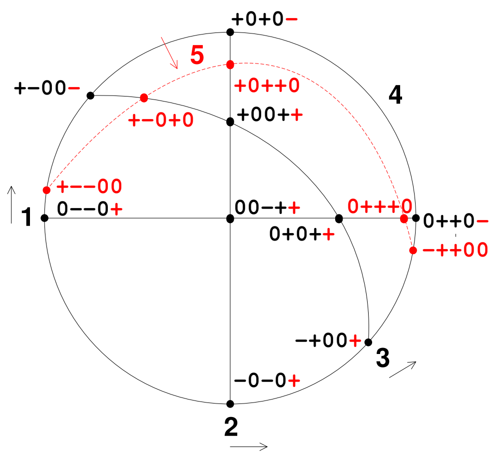

Figure 3.3: A covering extension of $\mathcal{M}_{3,4}$

We have n choices for the infinity hemisphere, $\binom{n-1}{r-2}$ possible pairs of antipodal vertices, $(r-2)!$ different ways of ordering the $(r-2)$ rotations, 2 directions per rotation, and finally 2 choices for the signature of the antipodal pair Y and $-Y$. Therefore, the total number of potential perturbations is $n\binom{n-1}{r-2}2^{r-2}(r-2)!\ 2 = n\binom{n-1}{r-2}2^{r-1}(r-2)!$.

Although the total number of perturbations grows fast, the actual number of covering extension is not that large since many perturbations yield the same combinatorial type of oriented matroid extension, see Table 3.1.

(r,n)	# of perturbations	# of covering extensions of $\mathcal{A}^*_{r-1,n-1}$
(3,5)	80	5
(3,6)	120	5
(3,7)	168	7
(3,8)	224	7
(4,6)	960	26
(4,7)	1680	44
(4,8)	2688	65

Table 3.1: Number of covering extension of cyclic arrangements

3.2 Computational results

The average diameter and number of external facets of the covering extensions of the cyclic arrangement are computed. The algorithms used are introduced in Chapter 4. The computational results disprove the hypothesis that $\Delta_{\mathcal{A}}(d,n)$, the maximum average diameter over all $\mathcal{A}_{d,n}$, is always achieved by a covering extension of the cyclic arrangement[2] $\mathcal{A}^*_{d,n-1}$.

3.2.1 Average diameter

Table 3.2 contrasts the value of $\Delta_{\mathcal{A}}(d,n)$ with the maximum average diameter $\Delta^c_{\mathcal{A}}(d,n)$ over all covering extensions of $\mathcal{A}^*_{d,n-1}$ as well as with $\delta(\mathcal{A}^*_{d,n})$, the average diameter of the cyclic arrangement $\mathcal{A}^*_{d,n}$.

(d,n)	$\Delta_{\mathcal{A}}(d,n)$	$\Delta^c_{\mathcal{A}}(d,n)$	$\delta(\mathcal{A}^*_{d,n})$
(3,6)	2	2	1.8
(3,7)	2.25	2.1	2
(3,8)	2.42	2.34...	2.14...
(3,9)	?	2.39...	2.25
(4,7)	2.2	2.2	2
(4,8)	2.71...	2.45...	2.28...

Table 3.2: Covering extensions $\Delta^c_{\mathcal{A}}(d,n)$ as a lower bound for $\Delta_{\mathcal{A}}(d,n)$

While the covering extensions of $\mathcal{A}^*_{d,n-1}$ fail to always reach $\Delta_{\mathcal{A}}(d,n)$, they provide a good lower bound. In particular, we obtain a new lower bound for $\Delta_{\mathcal{A}}(3,9)$. It was shown in [22] that $\Delta_{\mathcal{A}}(2,n) = \Delta^c_{\mathcal{A}}(2,n)$ and $\Delta_{\mathcal{A}}(3,6) = \Delta^c_{\mathcal{A}}(3,6)$. The computation shows that $\Delta_{\mathcal{A}}(4,7) = \Delta^c_{\mathcal{A}}(4,7)$, and all arrangements achieving $\Delta_{\mathcal{A}}(3,6)$ and $\Delta_{\mathcal{A}}(4,7)$ are covering arrangements.

3.2.2 Number of external facets

In addition to providing a good lower bound for $\Delta_{\mathcal{A}}(d,n)$, the covering extensions of specific types of $\mathcal{A}_{d,n-1}$ provide a good upper bound for $\Phi_{\mathcal{A}}(d,n)$, the minimum number of external facets over all $\mathcal{A}(d,n)$. Table 3.3 contrasts the value of $\Phi_{\mathcal{A}}(d,n)$ with the following values:

[2] One can check whether an arrangement $\mathcal{A}_{d,n}$ is a covering extension of $\mathcal{A}^*_{d,n-1}$ by removing the covering hyperplane and checking if the associated oriented matroid is alternating

(a) $\Phi^c_{\mathcal{A}}(d,n)$ – the minimum number of external facets over all covering extensions of $\mathcal{A}_{d,n-1}$ with $\phi(\mathcal{A}_{d,n-1}) = \Phi_{\mathcal{A}}(d,n-1)$.

(b) $\Phi^{c*}_{\mathcal{A}}(d,n)$ – the minimum number of external facets over all covering extensions of cyclic $\mathcal{A}^*_{d,n-1}$.

(c) $\phi(\mathcal{A}^*_{d,n})$ – the number of external facets of cyclic $\mathcal{A}^*_{d,n}$.

(d,n)	$\Phi_{\mathcal{A}}(d,n)$	$\Phi^c_{\mathcal{A}}(d,n)$	$\Phi^{c*}_{\mathcal{A}}(d,n)$	$\phi(\mathcal{A}^*_{d,n})$
(3,6)	22	22	22	24
(3,7)	32	32	34	40
(3,8)	44	44	48	60
(3,9)	?	58	64	84
(4,7)	47	47	47	50
(4,8)	84	84	88	100

Table 3.3: Covering extensions as an upper bound for $\Phi_{\mathcal{A}}(d,n)$

Table 3.3 shows that $\Phi^c_{\mathcal{A}}(d,n) = \Phi_{\mathcal{A}}(d,n)$ for the known entries, which leads to the following conjecture:

Conjecture 3.2.1 *the arrangements with* $\phi(\mathcal{A}_{d,n}) = \Phi_{\mathcal{A}}(d,n)$ *are covering extensions of the arrangements with* $\phi(\mathcal{A}_{d,n-1}) = \Phi_{\mathcal{A}}(d,n-1)$.

Table 3.4 lists the known entries for $\Phi_{\mathcal{A}}(d,n)$ as well as computationally achieved upper bounds believed to be tight. These upper bounds are indicated by a "$*$" sign and were obtained for given (d,n) by considering either: (i) all covering extensions of $\mathcal{A}_{d,n-1}$ satisfying $\phi(\mathcal{A}_{d,n-1}) = \Phi_{\mathcal{A}}(d,n-1)$ if $\Phi_{\mathcal{A}}(d,n-1)$ is known, or (ii) all covering extensions of $\mathcal{A}_{d,n-1}$ achieving the known minimum for $\phi(\mathcal{A}_{d,n-1})$. The values for $d=3$ lead to the following conjecture:

Conjecture 3.2.2 $\Phi_{\mathcal{A}}(3,n) = n^2 - 3n + 4$ *for* $n \geq 6$.

d \ n	3	4	5	6	7	8	9	10	11	12
2	3	6	8	10	12	14	16	18	20	22
3		4	12	22	32	44	58*	74*	92*	112*
4			5	20	45	84	130*	195*		
5				6	30	86*	184*			
6					7	42	142*			
7						8	56	218*		

Table 3.4: Known entries and upper bounds for $\Phi_{\mathcal{A}}(d, n)$

Chapter 4

Oriented Matroids Computation

A C++ package [60] is developed for oriented matroids computation to gain a better insight into the combinatorial structure of arrangement. It consists of classes for different representations of oriented matroids such as chirotopes, circuits/cocircuits and pseudo-sphere arrangement.

4.1 Exploring arrangements via oriented matroids computation

There are two advantages of using oriented matroids computation to explore the combinatorial structure of arrangements. First, all computations can be performed in a combinatorial way and hence numerical errors are avoided. Second, oriented matroids are enumerable [27, 28, 29]. The list of all nontrivial[1] uniform oriented matroids $\mathcal{M}_{r,n}$ for $r \leq 5$ and $n \leq 8$ as well as for $(r, n) = (3, 9)$ and $(3, 10)$ can be found in the online database [26]. The enumeration of uniform oriented matroids for $(d + 1, n + 1) = (4, 9)$ and $(5, 9)$ was recently performed by Finschi, Fukuda and Moriyama and will be uploaded on the online database [26].

The geometric properties of an arrangement such as the average diameter and the number of external facets can be computed from its corresponding uniform oriented matroid. As mentioned in Sections 1.3.3 and 2.2, although oriented matroid is an abstract combinatorial structure, we can interpret geo-

[1] Since the uniform $\mathcal{M}_{r,r+2}$ is unique, *nontrivial* means $n > r + 2$

metric notions such as faces, diameters and external facets through its pseudo-sphere arrangement representation.

In the pseudo-sphere arrangement representation of a uniform oriented matroid $\mathcal{M}_{r,n}$, every face can be labeled with a covector F of $\mathcal{M}_{r,n}$. In particular, the vertices are labeled with the cocircuits and the cells are labeled with the covectors not containing no 0's. The sign F_i, $i = 1, 2, \ldots, n$, indicates the position of the face relative to H_i, the pseudo-hyperplane corresponding to the ith pseudo-hemisphere. Face F is on the positive, negative side of or on H_i if $F_i = +, -$ or 0 respectively. Figure 4.1 shows all the covectors on the 4th pseudo-hemisphere.

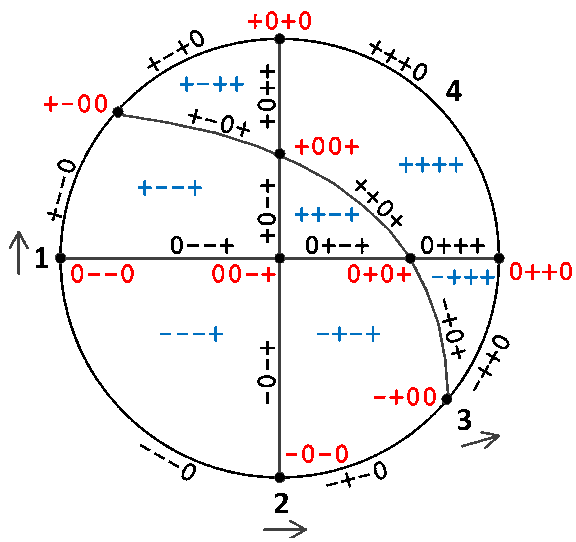

Figure 4.1: Covectors of $\mathcal{M}_{3,4}$ (the cocircuits are coloured in red)

4.1.1 Computation of the average diameter

Given the covector representing a cell, Algorithm 1 computes its diameter by constructing its skeleton graph. Two vertices X and Y are adjacent if X can be pivoted to Y, or vice versa, i.e. if there exists a pair of indices (i, j) with $1 \leq i, j \leq n$, $i \neq j$ such that: (i) $X_k = Y_k$ for all $k \neq i, j$, (ii) $X_i = 0$, $Y_i \neq 0$, and (iii) $X_j \neq 0$, $Y_i = 0$. The diameter of the cell is obtained by computing the diameter of the resulting skeleton graph.

Algorithm 1: CellDiameter

/* Compute the diameter of bounded cell C of a rank $d+1$ affine arrangement of $n+1$ pseudo-spheres represented by cocircuits $\mathcal{C}^*_{d+1,n+1}$. */
input : C, $\mathcal{C}^*_{d+1,n+1}$
output: δ ; /* diameter */

V $\leftarrow \phi$; /* vertices of cell C */
foreach $X \in \mathcal{C}^*_{d+1,n+1}$ **do**
  if X.ConformalTo (C) **then**
    V $\leftarrow$ V $\cup \{X\}$
G.AddVertices (V) ; /* skeleton graph of cell C */
foreach $(X \in V, Y \in V)$ **do**
  if Pivotable (X,Y) **then**
    G.AddEdge (X,Y)
return GraphDiameter (G)

Given an affine oriented matroid $(\mathcal{M}_{r,n}, k)$ in its cocircuits representation $(\mathcal{C}^*_{r,n}, k)$, where k is the infinity element. Algorithm 2 computes the average diameter of the bounded cells of $(\mathcal{M}_{r,n}, k)$. It uses several subroutines including Algorithm 1 for cell diameter, Algorithm 3 for covectors and Algorithm 4 for boundedness checking.

Algorithm 2: AverageDiameter

/* Compute the average diameter of the bounded cells of affine oriented matroid $(\mathcal{C}^*, k)$ (k is the infinity element). */
input : $(\mathcal{C}^*_{r,n}, k)$
output: δ ; /* average diameter */

$\mathcal{V}^*_{r,n} \leftarrow$ Covectors $(\mathcal{C}^*_{r,n})$; /* get the covectors */
NumBoundedCells $\leftarrow 0$;
Sum $\leftarrow 0$;
foreach $X \in \mathcal{V}^*_{r,n}$ **do**
  if NumZeros $(X) = 0$ **and** IsBounded $((\mathcal{C}^*_{r,n}, k), X)$ **then**
    NumBoundedCells $\leftarrow$ NumBoundedCells $+1$;
    Sum $\leftarrow$ Sum + CellDiameter (X) ;
return Sum/NumBoundedCells ;

Algorithm 3 computes the covectors from the cocircuits using repeated composition.

Algorithm 3: Covectors

/* Get the set of covectors $\mathcal{V}^*$ of oriented matroid $\mathcal{C}^*$ */
input : $\mathcal{C}^*_{r,n}$
output: $\mathcal{V}^*$

$\mathcal{V}^*_{r,n} \leftarrow \phi$;
$\mathcal{W}^*_{r,n} \leftarrow \mathcal{C}^*_{r,n}$;
while $|\mathcal{W}^*_{r,n}| > |\mathcal{V}^*_{r,n}|$ **do**
  $\mathcal{V}^*_{r,n} \leftarrow \mathcal{W}^*_{r,n}$;
  foreach $X \in \mathcal{V}^*_{r,n}$, $Y \in \mathcal{W}^*_{r,n}$ **do**
    $\mathcal{W}^*_{r,n} \leftarrow \mathcal{W}^*_{r,n} \cup \{X \circ Y\}$;
return $\mathcal{V}^*$;

Algorithm 4 checks the boundedness of a face by verifying that no vertex on the infinity is conformal to the covector of the face.

Algorithm 4: IsBounded

/* Check the boundedness of a face F in affine oriented matroid $(\mathcal{C}^*, k)$ (k is the infinity element). */
input : $(\mathcal{C}^*_{r,n}, k)$, F
output: **true** or **false**

foreach $X \in \mathcal{C}^*_{r,n}$ **do**
  if $X_k = 0$ **and** X.ConformalTo *(F)* **then**
    return false ;
return true ;

4.1.2 Computation of the number of external facets

For a uniform oriented matroid, the covector of a facet contains exactly one 0. A bounded facet is external if its corresponding covector is conformal to the covector of some unbounded cell. Algorithm 5 computes the number of external facets. It uses some of the subroutines defined in the previous section.

Algorithm 5: NumExternalFacets

/* Compute the number of external facets of affine oriented matroid $(\mathcal{C}^*, k)$ (k is the infinity element). */
input : $(\mathcal{C}^*_{r,n}, k)$
output: NumExternalFacets

$\mathcal{V}^*_{r,n} \leftarrow$ `Covectors` $(\mathcal{C}^*_{r,n})$; /* get the covectors */
/* Get the set of unbounded cells. */
$S \leftarrow \phi$;
foreach $X \in \mathcal{V}^*_{r,n}$ **do**
  if `NumZeros` $(X) = 0$ **and not** `IsBounded` $((\mathcal{C}^*_{r,n}, k), X)$ **then**
    $S = S \cup \{X\}$

/* Search for external facets. */
NumExternalFacets $\leftarrow 0$;
foreach $X \in \mathcal{V}^*_{r,n}$, $Y \in S$ **do**
  if `NumZeros` $(X) = 1$ **and** `IsBounded` $((\mathcal{C}^*_{r,n}, k), X)$ **and** X.`ConformalTo` (Y) **then**
    NumExternalFacets $\leftarrow$ NumExternalFacets $+1$

return NumExternalFacets ;

4.2 Computational results

4.2.1 Maximal average diameter

To determine the entries for $\Delta_{\mathcal{A}}(d, n)$, the maximum average diameter over all $\mathcal{A}_{d,n}$, for $d \leq 4$ and $n \leq 8$ we consider the set $\mathcal{M}_{d+1,n+1}$ of uniform oriented matroids. For each uniform oriented matroid, we iterates through the $n+1$ choices of setting one element as the infinity element, and compute the average diameters of the resulting affine oriented matroids. Finally, the realizability of the oriented matroids achieving $\Delta_{\mathcal{A}}(d, n)$ is checked. Computational results show that all oriented matroids with $\Delta_{\mathcal{A}}(d, n)$ turn out to be realizable and lead to the following question:

Question 4.2.1 *Can an affine non-realizable oriented matroid achieve the maximal average diameter?*

The entries for $\Delta_{\mathcal{A}}(d,n)$ with $d \leq 4$, $n \leq 8$ are listed in Table 4.1. The list of hyperplane arrangements satisfying $\delta(\mathcal{A}_{d,n}) = \Delta_{\mathcal{A}}(d,n)$ can be found in Appendix A.1 and in [60] in the form of affine chirotopes.

(d,n)	$\lvert\mathcal{M}_{d+1,n+1}\rvert$	$\Delta_{\mathcal{A}}(d,n)$
(2,5)	4	1.5
(2,6)	11	1.7
(2,7)	135	1.73...
(2,8)	4,382	1.80...
(3,6)	11	2
(3,7)	2,628	2.25
(3,8)	9,276,595	2.42...
(4,7)	135	2.2
(4,8)	9,276,595	2.71...

Table 4.1: $\Delta_{\mathcal{A}}(d,n)$ for $d \leq 4$ and $n \leq 8$

4.2.2 Minimal number of external facets

While computing the entries for $\Delta_{\mathcal{A}}(d,n)$, one can also obtain $\Phi_{\mathcal{A}}(d,n)$, the minimum number of external facets, i.e. facets belonging to exactly one bounced cell, over all $\mathcal{A}_{d,n}$. The computed entries for $\Phi_{\mathcal{A}}(d,n)$ are listed in Table 4.2. The list of hyperplane arrangements satisfying $\phi(\mathcal{A}_{d,n}) = \Phi_{\mathcal{A}}(d,n)$ can be found in Appendix A.2 and [60]. We recall that it was showed in [22] that $\Phi_{\mathcal{A}}(2,n) = 2(n-1)$.

(d,n)	$\lvert\lvert\mathcal{M}_{d+1,n+1}\rvert\rvert$	$\Phi_{\mathcal{A}}(d,n)$
(2,5)	4	8
(2,6)	11	10
(2,7)	135	12
(2,8)	4,382	14
(3,6)	11	22
(3,7)	2,628	32
(3,8)	9,276,595	44
(4,7)	135	47
(4,8)	9,276,595	84

Table 4.2: Entries for $\Phi_{\mathcal{A}}(d,n)$ for $d \leq 4$ and $n \leq 8$

The entry for $(d, n) = (3, 8)$ disproves Conjecture 2.3.1 stating that $\Phi_{\mathcal{A}}(d, n) \geq d\binom{n-2}{d-1}$ as we have $\Phi_{\mathcal{A}}(3, 8) = 44 < 45 = 3\binom{6}{2}$.

Remark 4.2.2 *All known affine oriented matroids minimizing the number of external facets are realizable.*

4.2.3 Maximal number of external facets

The maximal number of external facets is also computed to contrast the value with the number of external facets of the cyclic arrangements, see Table 4.3. Figure 4.2 illustrates the two arrangements of 6 lines that have the maximal number of external facets.

(d, n)	Maximal $\phi(\mathcal{A}(d, n))$	$\phi(\mathcal{A}^*(d, n))$
(2,6)	13	12
(2,7)	17	15
(3,6)	25	24
(3,7)	43	40
(3,8)	65	60
(4,7)	52	50
(4,8)	104	100

Table 4.3: Comparison of maximal $\phi(\mathcal{A}(d, n))$ and $\phi(\mathcal{A}^*(d, n))$

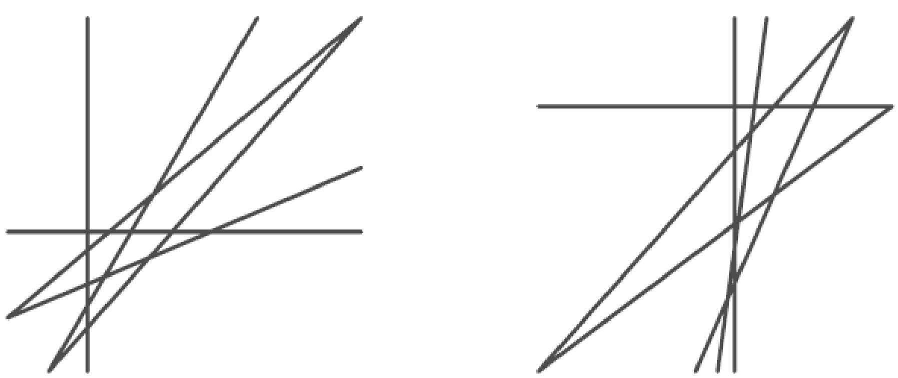

Figure 4.2: Two arrangements $\mathcal{A}_{2,6}$ with maximal $\phi(\mathcal{A}(2, 6))$

4.2.4 Simultaneously maximizing $\Delta_{\mathcal{A}}(d,n)$ and minimizing $\Phi_{\mathcal{A}}(d,n)$

The hypothesized relation between maximizing $\Delta_{\mathcal{A}}(d,n)$ and minimizing $\Phi_{\mathcal{A}}(d,n)$ is computationally substantiated by the existence for $d \leq 4$ and $n \leq 8$ of at least one simple arrangement simultaneously maximizing $\Delta_{\mathcal{A}}(d,n)$ and minimizing $\Phi_{\mathcal{A}}(d,n)$. The list of hyperplane arrangements satisfying both $\delta(\mathcal{A}_{d,n}) = \Delta_{\mathcal{A}}(d,n)$ and $\phi(\mathcal{A}_{d,n}) = \Phi_{\mathcal{A}}(d,n)$ can be found in Appendix A.3 and in [60].

The known entries for $\Delta_{\mathcal{A}}(d,n)$ are summarized in Table 4.4.

d \ $n-d$	2	3	4	5	6	$\dots$	$n-d$	$\dots$
2	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{17}{10}$	$\frac{26}{15}$	$\frac{38}{21}$	$\dots$	$2-\frac{2\lceil\frac{n}{2}\rceil}{(n-1)(n-2)}$	
3	$\frac{3}{2}$	2	$\frac{9}{4}$	$\frac{17}{7}$	$\geq\frac{67}{28}$			
4	$\frac{8}{5}$	$\frac{11}{5}$	$\frac{19}{7}$					
$\vdots$	$\vdots$							
d	$\frac{2d}{d+1}$							
$\vdots$	$\vdots$							

Table 4.4: Known entries for $\Delta_{\mathcal{A}}(d,n)$

Part II

Colourful Simplicial Depth and its Generalizations

Introduction

Given $d+1$ sets of points in dimension d such that the convex hulls of the $\mathbf{S}_i$'s contain p in their interior, the following Bárány's *Colourful Carathéodory Theorem* [4] shows that p must be contained in some *colourful simplex*, a simplex with each vertex from a different set.

We are interested in $\mu(d)$, the minimum number of colourful simplices drawn from $\mathbf{S}_1, \ldots, \mathbf{S}_{d+1}$ that contain $p \in R^d$ given that $p \in \text{int}(\text{conv}(\mathbf{S}_i))$ for each i, and assuming that the points in $\bigcup_i \mathbf{S}_i \cup \{p\}$ are in general position. Besides its intrinsic appeal, the *minimum colourful simplicial depth* $\mu(d)$ represents the minimum number of solutions to the colourful linear programming feasibility problem proposed by Bárány and Onn [6].

Deza et al [16] prove that $2d \le \mu(d) \le d^2+1$, that $\mu(d)$ is even for odd d, and that $\mu(2)=5$. The paper also conjectures that $\mu(d)=d^2+1$ for all $d \ge 1$. Subsequently, Bárány and Matoušek [5] verify the conjecture for $d=3$ and provide a lower bound of $\mu(d) \ge \max(3d, \left\lceil \frac{d(d+1)}{5} \right\rceil)$ for $d \ge 3$, while Stephen and Thomas [59] independently provide a stronger quadratic lower bound of $\mu(d) \ge \left\lfloor \frac{(d+2)^2}{4} \right\rfloor$. We prove that $\mu(d) \ge \lceil \frac{(d+1)^2}{2} \rceil$ for $d > 1$ in Chapter 5, and introduce a combinatorial generalization of colourful point configurations, octahedral systems, in Chapter 6.

Chapter 5

New Bound for the Minimum Colourful Simplicial Depth

The following Bárány's *Colourful Carathéodory Theorem* [4] shows that the minimum colourful simplicial depth $\mu(d)$ is at least 1.

Theorem 5.0.3 ([4]) *Let $\mathbf{S}_1, \mathbf{S}_2, \ldots, \mathbf{S}_{d+1}$ be finite sets of points in $\mathbb{R}^d$ such that $\mathbf{0} \in \mathrm{conv}(\mathbf{S}_i)$ for $i = 1 \ldots d+1$. Then there exists a set $S \subset \bigcup_i \mathbf{S}_i$ such that $|S \cap \mathbf{S}_i| = 1$ for $i = 1, \ldots, d+1$ and $\mathbf{0} \in \mathrm{conv}(S)$.*

In this chapter, we present a new lower bound of the minimum colourful simplicial depth and its proof [19]. This strengthens the previously known lower bound for all $d \geq 4$.

Theorem 5.0.4 *For $d \geq 1$, we have $\mu(d) \geq \lceil \frac{(d+1)^2}{2} \rceil$.*

5.1 Key observations

We call a set of points drawn from the $\mathbf{S}_i$'s *colourful* if it contains at most one point from each $\mathbf{S}_i$. We call a colourful set of d points which misses $\mathbf{S}_i$ an $\widehat{i}$*-transversal* . Note that $\widehat{i}$-transversals generate full dimensional pointed colourful cones; we say that a transversal *spans* a point if the point is contained in the associated cone. We have a colourful simplex containing $\mathbf{0}$ whenever the antipode of a point of colour i is spanned by an $\widehat{i}$-transversal. In Figure 5.1, the antipodes g_2, g_3 of G_2, G_3 are spanned by transversal $\{B_1, R_1\}$, thus yielding two colourful simplices $\{B_1, R_1, G_2\}$ and $\{B_1, R_1, G_3\}$.

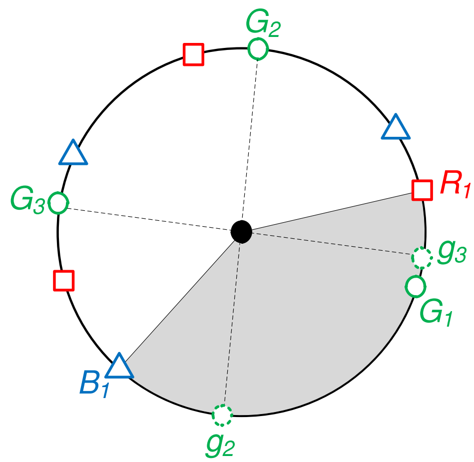

Figure 5.1: Transversal cone

A combinatorial *colourful octahedron*, or colourful cross polytope, is generated by a pair of disjoint $\widehat{i}$-transversals. Since each of the two transversals in the octahedron has d points of different colours, a octahedron contains 2^d transversals. An octahedron is said to span $\mathbb{S}^{d-1}$ if the 2^d transversals span $\mathbb{S}^{d-1}$.

The following *Octahedron Lemma* [5] states that every octahedron Ω either span $\mathbb{S}^{d-1}$, or every point in $\mathbb{S}^{d-1}$ that is spanned by colourful cones from Ω is spanned by at least two distinct such cones. In the case where the points of Ω form an octahedron in the geometric sense, these correspond to the cases where $\mathbf{0}$ is inside and outside Ω respectively. Figure 5.2 illustrates the two cases.

Lemma 5.1.1 (Octahedron lemma) *Let S, T be two disjoint transversals, and x be a point whose antipode is spanned by S.*

(i) If the colourful octahedron $S \cup T$ does not span $\mathbb{S}^{d-1}$, then there exists a transversal $T' \in S \cup T$, $T' \neq S$, that also spans the antipode of x.

(ii) If S is the unique transversal in $S \cup T$ spanning the antipode of x, then $S \cup T$ spans $\mathbb{S}^{d-1}$.

Our strategy for finding distinct colourful simplices is to begin with a transversal that generates at least one colourful simplex, and get further points

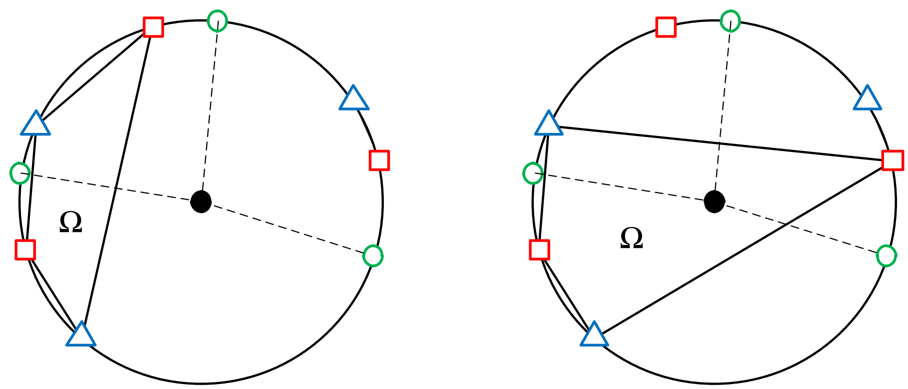

Figure 5.2: Two illustrations of Octahedron lemma

from octahedra that include this transversal. We consider different cases based on the number of colourful simplices generated by the initial transversal, and on how many of the octahedra span $\mathbb{S}^{d-1}$.

5.2 Proof of Theorem 5.0.4

The Colourful Carathéodory Theorem proves the existence of at least one colourful simplex containing $\mathbf{0}$. It gives an antipode of colour $(d+1)$ that lies in the cone generated by a $\widehat{d+1}$-transversal T. Without loss of generality we can number the points of $\mathbf{S}_1, \mathbf{S}_2, \ldots, \mathbf{S}_d$ so that point $(d+1)$ of S_i is included in T. The remaining points of the S_i's can be numbered arbitrarily. Let T_i be the set that contains the points numbered i from $\mathbf{S}_1, \mathbf{S}_2, \ldots \mathbf{S}_{d+1}$. Then each T_i is a $\widehat{d+1}$-transversal and $T_{d+1} = T$. Further, the sets $T_1, T_2, \ldots, T_{d+1}$ are pairwise disjoint. Let L be the set of antipodes of colour $(d+1)$ spanned by T_{d+1}, where $|L| = l > 0$. Figure 5.3 illustrates the way the transversals are labeled.

5.2.1 Points from d octahedra that share a transversal

Now consider the d octahedra $\Omega_1, \Omega_2, \ldots, \Omega_d$ given by pairing T_i with T_{d+1} for $i = 1, 2, \ldots, d$. Except for the common transversal T_{d+1}, every $\widehat{d+1}$-transversal found among the Ω_i's is distinct. Suppose that b of the octahedra span $\mathbb{S}^{d-1}$. We count the colourful simplices containing $\mathbf{0}$ in the following way:

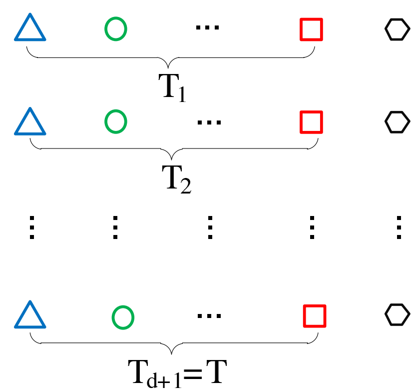

Figure 5.3: Labeling of the transversals

(i) There are l antipodes of colour $(d+1)$ spanned by T_{d+1}, yielding l simplices containing $\mathbf{0}$.

(ii) There are $(d+1-l)$ antipodes of colour $(d+1)$ that are not spanned by T_{d+1}, and hence must be spanned by a different transversal from each of the b sphere-spanning octahedra in $\{\Omega_i\}$. This gives us $b(d+1-l)$ distinct simplices containing $\mathbf{0}$.

(iii) There are $(d-b)$ octahedra in $\{\Omega_i\}$ that do not span $\mathbb{S}^{d-1}$. By the Octahedron Lemma, each of the l antipodes spanned by T_{d+1} must also be spanned by a second transversal from the octahedron generated by T_{d+1} and T_i. So we find an additional $(d-b)l$ distinct simplices along with the l simplices generated by the antipodes with T_{d+1} itself.

In total we have $l + b(d+1-l) + (d-b)l = (d+1)(b+l) - 2bl$ distinct colourful simplices containing $\mathbf{0}$.

5.2.2 Case study

In the above argument, T_{d+1} can be any $\widehat{d+1}$-transversal containing an antipode of colour $(d+1)$. Note that if each antipode of colour $(d+1)$ is spanned by at least j $\widehat{d+1}$-transversals, then depth$(\mathbf{0}) \geq j(d+1)$. We can take T_{d+1} to be a transversal spanning the least covered antipode. As we move through the possible values of i in the argument of Section 5.2.1, whenever the octahedron fails to cover $\mathbb{S}^{d-1}$ we will see a new cone covering the lightly spanned antipode. Hence $(j-1)+b \geq d$. We thus have

$$\begin{aligned} \text{depth}(\mathbf{0}) \geq \max\{j(d+1), (d+1)(b+l) - 2bl\} \\ j \geq 1, b \geq 1, l \leq d \\ j + b \geq d + 1 \end{aligned} \tag{5.2.1}$$

We consider two cases.

Case 1: $l \leq \frac{d+1}{2}$

When $l \leq \frac{d+1}{2}$, by taking either $j \geq \frac{d+1}{2}$ or $b \geq \frac{d+2}{2}$ we get $\text{depth}(\mathbf{0}) \geq \frac{d^2+2d+1}{2}$ from Equation 5.2.1.

Case 2: $l \geq \frac{d+2}{2}$

In this case, we begin with l simplices containing $\mathbf{0}$ differing only in the $(d+1)$st colour. We can repeat this exercise for each colour, in which case we will either find that for each colour i, $l_i \geq \frac{d+2}{2}$, or, for some colour i, $l_i \leq \frac{d+1}{2}$. In the latter case, we apply the analysis above to get at least $\frac{d^2+2d+1}{2}$ distinct simplices containing zero.

If it happens that we get $l_i \geq \frac{d+2}{2}$ for each i, then for each i we have a set L_i of at least l_i antipodes of colour i which lie on a single $\widehat{i}$-transversal U_i. These generate $(d+1)$ sets $X_1, X_2, \ldots X_{d+1}$ of at least $l = \min_i(l_i) \geq \frac{d+2}{2}$ colourful simplices. There may be some duplication between sets, but we note that the simplices within each set are distinct and differ only in the ith colour.

We can identify the simplices that make up the X_i's with vectors in $\{1, 2, \ldots, d+1\}^{d+1}$. We find it helpful to consider them as vectors in $\mathbb{R}^{d+1}$ unrelated to the initial configuration.

A simplex α_d belonging to a given X_i is represented by a vector in $\mathbb{R}^{d+1}$ in the following way. The axes correspond to the $d+1$ colours, and the qth coordinate is set to the index in S_q of the point of colour q of α_d. We recall that the index of points in S_q is set by the arbitrary numbering of points of colour q.

The vectors associated to the simplices from a given X_i lie on a line segment in the ith coordinate direction. If a simplex is in both X_i and X_q, then the associated vector must lie at the intersection of the corresponding line segments.

Lemma 5.2.1 *There are at most d duplicate vectors in the union of the X_i's, where a vector that is in $k+1$ sets is counted as k duplicate vectors.*

Proof. Consider adding the sets repeatedly. We will say that two sets are in the same *component* if they contain a common point, and extend this to an equivalence relation. We remark that each component is contained in the topological component formed by taking the union of the line segments associated to the X_i's, but a given topological component will contain multiple components if the points of intersection of the line segments are not included in the corresponding X_i's.

We begin with $c = 0$ components and $k = 0$ duplicate vectors. Each added set either creates a new component or intersects r components, producing r duplicate vectors while reducing the number of components by $(r-1)$ through the equivalence relation. Therefore at each step $c+k$ increases by 1. Upon termination, we will have at least 1 component, and hence at most d duplicate vectors. □

Then the X_i's contain distinct simplices except possibly for up to $d+1-c \leq d$ duplicates arising in this construction, where c is the number of components. This gives us a total of $(d+1)l-(d+1-c) = (d+1)(l-1)+c$ distinct simplices containing $\mathbf{0}$.

However, if c is small, we can readily find additional distinct simplices containing $\mathbf{0}$ by observing that for a fixed colour i, for instance one attaining $l = l_i$, we also have $(d+1-l)$ antipodes outside of L_i. Each of these antipodes must generate some colourful simplex containing $\mathbf{0}$. In fact, for each antipode omitted, we could get $\frac{d+1}{2}$ simplices since either l_i or b is this large, but it does not improve our worst case. Call this set of simplices M, and again consider them as vectors in $\mathbb{R}^{d+1}$. They are not included among the vectors associated to simplices in X_i, since they have different values of coordinate i.

The vectors associated to simplices in M could duplicate vectors from components other than the one containing X_i. However, each such component has a fixed value of colour i. If $c-1 \geq d+1-l$ it may be the case that all such simplices are duplicates, but in that case $(d+1)(l-1)+c \geq dl+1$. If $c-1 < d+1-l$ we get at least $d+2-l-c$ additional distinct simplices from vertices omitted from the $(d+1)$ sets. This again guarantees us at least $(d+1)(l-1)+c+(d+2-l-c) = dl+1$ distinct simplices.

Now as $l \geq \frac{d+2}{2}$ we get at least $d\frac{d+2}{2}+1 = \frac{d^2+2d+2}{2}$ distinct simplices containing $\mathbf{0}$. Thus our overall worst case for this analysis is at $\frac{d^2+2d+1}{2} = \frac{(d+1)^2}{2}$, which can be rounded up to an integer when d is even. This improves the known bounds for $d \geq 4$, in particular from 12 to 13 when $d = 4$. We remark that unlike previous general approaches, this analysis gives the tight bound of 5 when $d = 2$.

5.3 Generalizations of the Colourful Carathéodory Theorem

5.3.1 A first generalization of the Colourful Carathéodory Theorem

In this subsection, general position is assumed, and in particular we assume that $|S_i| = d+1$ for $i = 1, \ldots, d+1$. Note that Theorems 5.3.1 and 5.3.5 do require general position. The Colourful Carathéodory Theorem was generalized by Arocha et al. [1] and by Holmsen et al. [39] who provided a more general sufficient condition for the existence of a colourful simplex containing the origin.

Theorem 5.3.1 ([1, 39]) *Let* $\mathbf{S}_1, \mathbf{S}_2, \ldots, \mathbf{S}_{d+1}$ *be finite sets of points in* $\mathbb{R}^d$ *such that* $\mathbf{0} \in \mathrm{conv}(\mathbf{S}_i \cup \mathbf{S}_j)$ *for* $1 \leq i < j \leq d+1$. *Then there exists a set* $S \subset \bigcup_i \mathbf{S}_i$ *such that* $|S \cap \mathbf{S}_i| = 1$ *for* $i = 1, \ldots, d+1$ *and* $\mathbf{0} \in \mathrm{conv}(S)$.

As an analog to $\mu(d)$, we define $\mu^\diamond(d)$ as the minimum number of colourful simplices containing $\mathbf{0}$ in any configurations satisfying the condition of Theorem 5.3.1 and the general position assumption. Note that, without the general position assumption, the corresponding minimum number of colourful simplices containing $\mathbf{0}$ is at most $\min_i |S_i|$.

Proposition 5.3.2 $\mu^\diamond(d) \leq d+1$.

Proof. We construct a configuration that has exactly $d+1$ colourful simplices. We use the terms for locations on $\mathbb{S}^d$ as defined in [16] such as north/south polar region, tropic of Cancer and tropic of Capricorn, as shown in Figure 5.4. The points of each colour except colour $d+1$ are put into the above locations in the same way as [16] – one in tropic of Cancer, one in tropic of Capricorn and the others in north polar region such that the origin is contained in the convex hull. This way, the region around the south pole is spanned by only one colourful cone, which consists of the points located in the tropic of Capricorn. Then we make all antipodes of colour $d+1$ in the south polar region by putting all points of colour $d+1$ in the north polar region, see Figure 5.5.

Since $\mathbf{0}$ is in $\text{conv}(\mathbf{S}_i)$ for $i = 1, 2, \ldots, d$, the relaxed conditions are satisfied. Also, each colour from $1, \ldots, d$ has a unique point which is a generator for all $(d+1)$ colourful simplices containing $\mathbf{0}$, because the antipodes of colour $d+1$ are in a region that is spanned by only one colourful cone. □

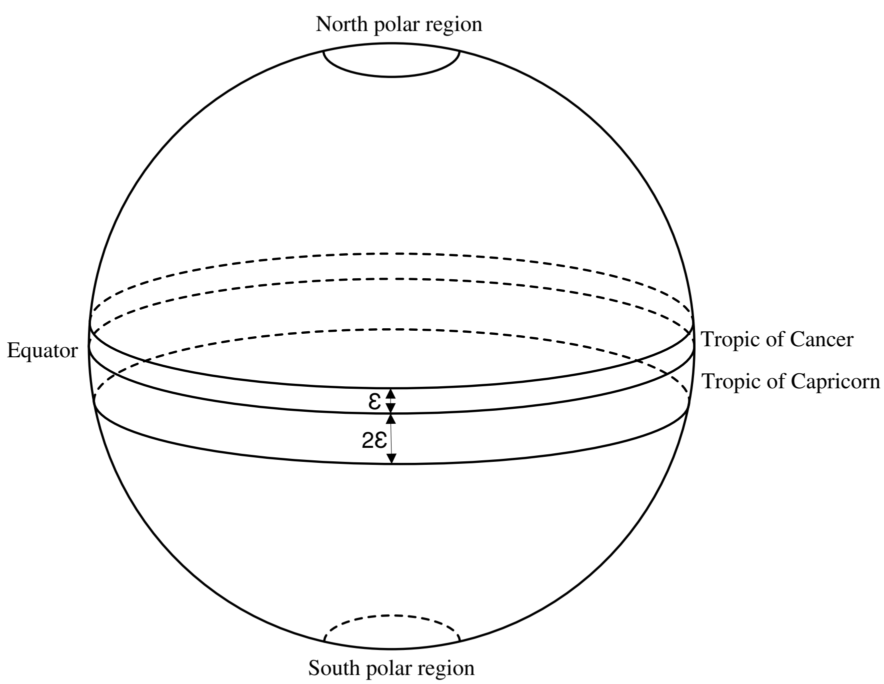

Figure 5.4: Sphere locations

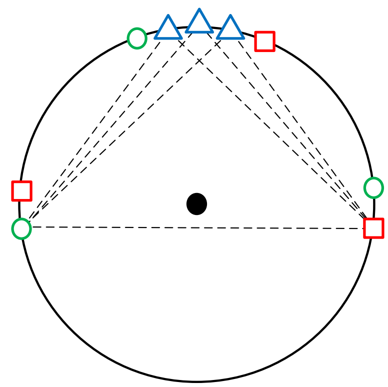

Figure 5.5: A configuration in dimension 2 satisfying the condition of Theorem 5.3.1 and with the origin having a colourful simplicial depth of 3

Remark 5.3.3 *In contrast to the situation when* $\mathbf{0}$ *is in all the* $\mathbf{S}_i$*'s, most points from the* $\mathbf{S}_i$ *generate no colourful simplices containing* $\mathbf{0}$.

Proposition 5.3.4 $\mu^{\Diamond}(2) = 3$.

Proof. We show that any configuration in dimension 2 has at least three colourful simplices. The condition $\mathbf{0} \in \text{conv}(\mathbf{S}_1 \cup \mathbf{S}_2)$ indicates that every half-circle contains a point from $\mathbf{S}_1 \cup \mathbf{S}_2$. If the circle is covered by colourful cones, then each antipode of the remaining colour generates a colourful simplex containing $\mathbf{0}$ and we are done. Otherwise, some segment of the circle is not covered by any colourful cone. This segment must be bounded by two points p and p' of the same $\mathbf{S}_i$, say $\mathbf{S}_1$. The three points of $\mathbf{S}_2$ then are on the longer arc between these points, and for each point of $\mathbf{S}_2$, every point on the longer arc is covered by a colourful cone using that point and either p or p'. The condition that $\mathbf{0} \in \text{conv}(\mathbf{S}_2 \cup \mathbf{S}_3)$ forces at least one of the antipodes of $\mathbf{S}_3$ to lie in the arc that spans the three points of $\mathbf{S}_2$.

We have shown that $\mu^{\Diamond}(2) \geq 3$. Since $\mu^{\Diamond}(2) \leq 3$ by Proposition 5.3.2, we have $\mu^{\Diamond}(2) = 3$. □

5.3.2 A further generalization of the Colourful Carathéodory Theorem

Meunier and Deza [46] further generalized the sufficient condition for the existence of a colourful simplex containing the origin. Let $\overrightarrow{x_k\mathbf{0}}$ denote the ray originating from x_k towards $\mathbf{0}$.

Theorem 5.3.5 ([46]) *Let $\mathbf{S}_1, \mathbf{S}_2, \ldots, \mathbf{S}_{d+1}$ be finite sets of points in $\mathbb{R}^d$. Assume that, for each $1 \le i < j \le d+1$, there exists $k \notin \{i, j\}$ such that, for all $x_k \in \mathbf{S}_k$, the convex hull of $\mathbf{S}_i \cup \mathbf{S}_j$ intersects the ray $\overrightarrow{x_k\mathbf{0}}$ in a point distinct from x_k. Then there exists a set $S \subset \bigcup_i \mathbf{S}_i$ such that $|S \cap \mathbf{S}_i| = 1$ for $i = 1, \ldots, d+1$ and $\mathbf{0} \in \mathrm{conv}(S)$.*

Up to an additional argument to handle degenerate configurations, Theorem 5.3.5 can be derived from the slightly stronger Theorem 5.3.6 where $H^+(T_i)$ denotes, for any $\widehat{i}$-transversal T_i, the open half-space defined by $\mathrm{aff}(T_i)$ and containing $\mathbf{0}$.

Theorem 5.3.6 ([46]) *Let $\mathbf{S}_1, \mathbf{S}_2, \ldots, \mathbf{S}_{d+1}$ be finite sets of points in $\mathbb{R}^d$ such that the points in $\bigcup_i \mathbf{S}_i \cup \{\mathbf{0}\}$ are distinct and in general position. Assume that, for any $i \neq j$, $(\mathbf{S}_i \cup \mathbf{S}_j) \cap H^+(T_i) \neq \emptyset$ for any $\widehat{i}$-transversal T_i. Then there exists a set $S \subset \bigcup_i \mathbf{S}_i$ such that $|S \cap \mathbf{S}_i| = 1$ for $i = 1, \ldots, d+1$ and $\mathbf{0} \in \mathrm{conv}(S)$.*

Note that, as the conditions Theorem 5.0.3 and Theorem 5.3.1, but unlike the one of Theorem 5.3.6, the condition of Theorem 5.3.5 is computationally easy to check. In the plane and assuming general position, Theorem 5.3.5 can be generalized to Theorem 5.3.7.

Theorem 5.3.7 ([46]) *Let $\mathbf{S}_1, \mathbf{S}_2, \mathbf{S}_3$ be finite sets of points in $\mathbb{R}^2$ such that the points in $\mathbf{S}_1 \cup \mathbf{S}_2 \cup \mathbf{S}_3 \cup \{\mathbf{0}\}$ are distinct and in general position. Assume that, for pairwise distinct $i, j, k \in \{1, 2, 3\}$, the convex hull of $\mathbf{S}_i \cup \mathbf{S}_j$ intersects the line $\mathrm{aff}(x_k, \mathbf{0})$. Then there exists a set $S \subset \mathbf{S}_1 \cup \mathbf{S}_2 \cup \mathbf{S}_3$ such that $|S \cap \mathbf{S}_i| = 1$ for $i = 1, 2, 3$ and $\mathbf{0} \in \mathrm{conv}(S)$.*

5.4 A Combinatorial Generalization

The methods in Section 5.2 rely on the combinatorial structure of the vectors representing the simplices. Indeed, there is a nice generalization of the colourful simplicial depth problem to systems of vectors in $\{1, 2, \ldots, d+1\}^{d+1}$.

Given sets of points $\mathbf{S}_1, \ldots, \mathbf{S}_{d+1}$, we form the system of vectors $\mathcal{V}$ where $\mathbf{v} = (s_1, \ldots, s_{d+1})$ is in $\mathcal{V}$ exactly if the colourful simplex described by $\mathbf{v}$ contains $\mathbf{0}$. In this context, $\widehat{i}$-*transversals* are simply vectors with the ith coordinate removed, and *octahedra* are pairs of disjoint $\widehat{i}$-transversals. The system $\mathcal{V}$ satisfies certain properties corresponding to Bárány's Colourful Carathéodory Theorem and the Octahedron lemma.

This combinatorial generalization, which we call octahedral systems, will be further discussed in Chapter 6.

Chapter 6

Octahedral Systems

We are interested in the set systems of the following type: the base set $\mathbf{S}$ is partitioned into *colours* $\mathbf{S}_1, \mathbf{S}_2, \ldots \mathbf{S}_m$ for some m, and the sets consist of one element from each $\mathbf{S}_i$. In other words, these are m-uniform hypergraphs where each hyperedge has a unique intersection with each colour $\mathbf{S}_i$. We will sometimes refer to the sets that belong to a given system as *edges*. We call a subset of $\mathbf{S}$ *colourful* if it contains at most one point from each $\mathbf{S}_i$. Thus the edges of any octahedral system are colourful.

We call a colourful set of d points which misses $\mathbf{S}_i$ an $\widehat{i}$*-transversal*, and call any pair of disjoint $\widehat{i}$*-transversals* an *octahedron*. We say that an m-uniform collection of colourful edges forms an *octahedral system* if it satisfies the following property:

Property 6.0.1 *For any octahedron Ω consisting of two $\widehat{i}$-transversals, the parity of the set of edges using points from Ω and a fixed point s_i for the* ith *coordinate is the same for all choices of s_i.*

Figure 6.1 demonstrates an octahedral system of three colours ($m = 3$). If we label the points from top down starting from 1, the edges of the system are $(1,1,1), (1,1,2), (1,2,1), (1,2,2), (1,3,3)$. If the last edge is absent, Property 6.0.1 is not satisfied any more. As shown in Figure 6.2, there is one edge using G_1 and points from Ω, but none using G_2 and G_3.

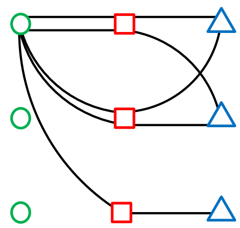

Figure 6.1: An octahedral system with 3 colours ($m = 3$)

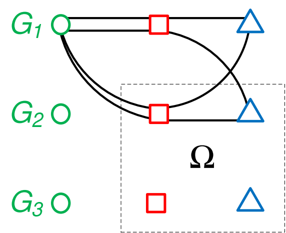

Figure 6.2: A non-octahedral system

6.1 Octahedral systems and colourful simplicial depth

The octahedral system is a combinatorial generalization of the colourful point configuration arising from the study of colourful simplicial depth as discussed in Chapter 5. From any such colourful point configuration, we can form a system of vectors $\mathcal{V}$ where $\mathbf{v} = (s_1, \ldots, s_{d+1})$ is in $\mathcal{V}$ if and only if the colourful simplex described by $\mathbf{v}$ contains $\mathbf{0}$. Such a system satisfies Property 6.0.1, which is implied by the *Octahedron Lemma*.

When the points of Ω are drawn from $\mathbb{R}^d$ and form an octahedron in the geometric sense, i.e. conv(Ω) is an octahedron and same coloured points are not adjacent in the skeleton of the polytope, then the odd, respectively even, case corresponds to $\mathbf{0}$ lying inside, respectively outside, Ω. Figure 5.2 illustrates this in a two dimensional case where $\mathbf{0}$ is at the centre of a circle that contains points of the three colours.

We can get lower bounds for the number of colourful simplices containing p for given configurations by showing that certain small octahedral systems cannot exist. In particular, it leads to two interesting combinatorial questions:

what is the smallest non-empty octahedral system in terms of the number of edges on $(d+1)$ sets of $(d+1)$ points, and what is the smallest such system where every point is contained in some edge. In Section 6.3 we show that the answer to the first question is $d+1$ and use this to prove a conjecture about point configurations. The second question suggests a method of computationally attacking the colourful simplicial depth problem, see below, at least for small dimension.

6.2 Octahedral Problems

The strong version of Bárány's Colourful Carathéodory Theorem says that when a colourful configuration satisfies $\mathbf{0} \in \text{conv}(S_i)$ for $i = 1, \ldots, d+1$, then every point in $\mathbf{S}$ is part of some colourful simplex. Thus the octahedral system generated by such a colourful configuration must satisfy:

Property 6.2.1 *Every element of* $\{1, 2, \ldots, d+1\}$ *is in some* $\mathbf{v} \in \mathcal{V}$.

In particular, any colourful configuration satisfying with $\mathbf{0} \in \text{conv}(S_i)$ for $i = 1, \ldots, d+1$ must generate a system $\mathcal{V}$ satisfying Property 6.0.1 and Property 6.2.1. For example, the low colourful simplicial depth configurations of [16] generate such a system with $(d+1)$ sets of $(d+1)$ points, containing (d^2+1) vectors. We define $\nu(d)$ to be the minimum number of vectors in an octahedral system of $(d+1)$ points in $(d+1)$ colours satisfying Properties 6.0.1 and 6.2.1, and $\nu^\diamondsuit(d)$ to be the minimum number of vectors of an similar system satisfying Property 6.0.1 only. Then we have $\nu(d) \leq \mu(d) \leq d^2+1$ and $\nu^\diamondsuit(d) \leq \mu^\diamondsuit(d) \leq d+1$. In Section 6.3 we show that $\nu^\diamondsuit(d) = \mu^\diamondsuit(d) = d+1$. In Section 6.4 we show that $\nu(d) = d^2+1$ for $d = 2, 3$, and conjecture that it may hold for all d. In particular, computation of $\nu(d)$ for small d gives us a finite procedure that can prove interesting lower bounds for $\mu(d)$.

Remark 6.2.2 *In [16] it was observed that* $\mu(d)$ *is even for odd* d*. Similarly it is easy to see that when* $m = d+1$ *is even, all octahedral systems have an even number of vectors. In particular, both* $\nu(d)$ *and* $\nu^\diamondsuit(d)$ *are even for odd* d*.*

6.3 Proof that $\mu^\Diamond(d) = d + 1$

Consider the octahedral system corresponding to the configuration provided in the proof of Proposition 5.3.2 and showing that $\mu^\Diamond(d) \leq d+1$ for $d \geq 2$. This octahedral system consists of the $d+1$ edges $\{(1,1,\ldots,i) : i = 1,2,\ldots,d+1\}$ and therefore we have $\nu^\Diamond(d) \leq d+1$ for $d \geq 2$. See Figure 6.3 for an illustration of minimal octahedral system in dimension 2. In this section we show that any non-empty octahedral system of $(d+1)$ sets of $(d+1)$ points has at least $(d+1)$ vectors, and hence that $\nu^\Diamond(d) = \mu^\Diamond(d) = d+1$.

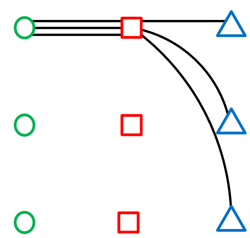

Figure 6.3: An octahedral system with $d+1$ edges for $d = 2$

Proposition 6.3.1 *For any $d \geq 2$, we have $\nu^\Diamond(d) = \mu^\Diamond(d) = d+1$.*

Proof. Assume that there is an octahedron Ω consisting of two $\widehat{i}$-transversals and a point $s \in S_i$ such that there are an odd number of edges using points from Ω and s. Then it follows immediately from Property 6.0.1 that there is at least one edge that uses points from Ω and any point in S_i. Therefore $\nu^\Diamond(d) \geq d+1$ as $|S_i| = d+1$.

Assume then that there exists no such octahedron with odd parity, but that the system contains some edge E. We view E as being formed by a $\widehat{i}$-transversal T and a point $s \in S_i$ and generate edges in the following way: Consider the d disjoint $\widehat{i}$-transversals T_i for $i = 1,2,\ldots,d$ generated from the remaining points, and the d octahedra $\Omega_1, \Omega_2, \ldots, \Omega_d$ given by pairing T_i with T for $i = 1,2,\ldots,d$. For each i, besides E, there is at least one other edge that uses s and the points from Ω_i due to the even parity. Therefore $\nu^\Diamond(d) \geq d+1$.

In both cases $\nu^\Diamond(d) \geq d+1$. Thus we have $\nu^\Diamond(d) = \mu^\Diamond(d) = d+1$ as $\nu^\Diamond(d) \leq \mu^\Diamond(d) \leq d+1$. □

Remark 6.3.2 *Meunier and Deza [46] recently proved that any condition that guarantees the existence of at least 1 colourful simplex containing the origin, actually guarantees that at least* $\min_i |S_i|$ *such colourful simplices exist. In particular, assuming general position implies that* $|S_i| = d+1$ *for* $i = 1, \ldots, d+1$*, and therefore we have at least* $d+1$ *such colourful simplices.*

6.4 Computational Approach

For a given d, the computational approach consists of ruling out a given value k for $\nu(d)$ via an exhaustive computer search showing that no system $\mathcal{V}$ of size k can satisfy Property 6.0.1 and Property 6.2.1. This approach was used to show that $\nu(2) > 3$ and $\nu(3) > 8$. In other words, this approach verifies computationally that $\nu(2) = \mu(2) = 5$ and $\nu(3) = \mu(3) = 10$ using the fact that $\nu(3)$ must be even, see Remark 6.2.2.

In this section we propose ways to normalize the vector system which significantly speed up the enumeration. We also present a constraint programming formulation of the problem.

6.4.1 Normalization of the vector system

Recolouring and relabelling of the points do not change the combinatorics of the point configuration. This symmetry will result in many duplicates in enumeration. In order to speed up the enumeration of vector systems for $\nu(d)$ we normalize the vector system in the following ways.

(N1) First, since the vector system $\mathcal{V}$ is not empty, we can assume vector $(0, 0, \ldots, 0) \in \mathcal{V}$.

(N2) If there is a *covering* octahedron, i.e. one that generates an odd number of vectors for each point of the excluded colour, we can take the excluded colour to be the final one, an octahedron of the system to be $\{(0, \ldots, 0), (1, \ldots, 1)\}$, with the labelings of the points of colours $1, \ldots, d$ chosen so that (N1) is satisfied.

A Python routine that searches for small octahedral systems using these normalizations is available at [61].

6.4.2 Constraint programming approach

The other computational approach for $\nu(d)$ is to exploit the fact that there is a sphere covering octahedron for each missing colour and model the search for a valid vector system as a constraint programming problem.

We can start with the following collection of vectors $\mathcal{V}^{\circ}$. Each block of $(d+1)$ vectors represents the simplices derived from a sphere covering octahedron for a missing colour.

$$\begin{gathered}(1, x_{1,1}^2, x_{1,1}^3, \ldots, x_{1,1}^{d+1}),\ (2, x_{1,2}^2, x_{1,2}^3, \ldots, x_{1,2}^{d+1}),\ \ldots,\\ (d+1, x_{1,d+1}^2, x_{1,d+1}^3, \ldots, x_{1,d+1}^{d+1});\\ (x_{2,1}^1, 1, x_{2,1}^3, \ldots, x_{3,1}^{d+1})\ ,\ (x_{2,2}^1, 2, x_{2,2}^3, \ldots, x_{2,2}^{d+1}),\ \ldots,\\ (x_{2,d+1}^1, d+1, x_{2,d+1}^3, \ldots, x_{2,d+1}^{d+1});\\ \cdots\\ (x_{d+1,1}^1, \ldots, x_{d+1,1}^d, 1),\ (x_{d+1,2}^1, \ldots, x_{d+1,2}^d, 2),\ \ldots,\ (x_{d+1,d+1}^1, \ldots, x_{d+1,d+1}^d, d+1).\end{gathered}$$

The domain of each variable is $\{1, 2, \ldots, d+1\}$. Then we have a constraint programming satisfaction problem: Given a value k, find an assignment of values to the variables such that $|\mathcal{V}^{\circ}| \leq k$ and the following constraints are satisfied:

(1) $x_{1,1}^i = 1$ for all i and $x_{1,j}^i \in \{1, 2\}$ for all i and $j \geq 2$. These constraints are derived from the normalizations of the vector system as discussed in subsection 6.4.1.

(2) $|\{x_{j,1}^i, x_{j,2}^i, \ldots, x_{j,d+1}^i\}| \leq 2$ for all i and j because they are from an octahedron.

(3) Constraints corresponding to Property 6.0.1.

If no solution is found, then $\nu(d) \neq k$.

6.5 Further questions

We can ask some further questions of interest:

Question 6.5.1 *Can all octahedral systems of* $(d+1)$ *sets of* $(d+1)$ *points be obtained as the vectors of point configurations in* $\mathbb{R}^d$*, and can all such configurations covering all points can be obtained as the vectors of configurations with a given condition?*

Question 6.5.2 *How many octahedral systems and covering octahedral systems are there for a given* m*? We remark that for* $m = 1$ *we have 2 systems, 1 of which is covering, and for* $m = 2$ *we have 8 and 3; if we count only up to isomorphism these numbers are 4 and 2 respectively.*

Appendix A

Computational results

In the following lists, a hyperplane arrangement $\mathcal{A}_{d,n}$ is represented by the affine chirotope $(\chi_{d+1,n+1}, n+1)$ of its corresponding affine oriented matroid. The signs of the chirotope are ordered reverse lexicographically and the infinity element is always $(n+1)$, the last element.

A.1 List of arrangements with maximum average diameter

The following tables list the arrangements with $\Delta_{\mathcal{A}}(d,n)$ for $(d,n) = (3,7)$, $(3,8)$, $(4,7)$ and $(4,8)$.

Table A.1: $\mathcal{A}_{3,7}$ with $\delta(\mathcal{A}_{3,7}) = \Delta_{\mathcal{A}}(3,7) = \frac{9}{4}$

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Table A.2: $\mathcal{A}_{3,8}$ with $\delta(\mathcal{A}_{3,8}) = \Delta_{\mathcal{A}}(3,8) = \frac{17}{7}$

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Table A.3: $\mathcal{A}_{4,7}$ with $\delta(\mathcal{A}_{4,7}) = \Delta_{\mathcal{A}}(4,7) = \frac{11}{5}$

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Table A.4: $\mathcal{A}_{4,8}$ with $\delta(\mathcal{A}_{4,8}) = \Delta_{\mathcal{A}}(4,8) = \frac{19}{7}$

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A.2 List of arrangements with minimum number of external facets

The following tables list the arrangements with $\Phi_{\mathcal{A}}(d,n)$ for $(d,n) = (3,7)$, $(3,8)$, $(4,7)$, and $(4,8)$.

Table A.5: $\mathcal{A}_{3,7}$ with $\phi(\mathcal{A}_{3,7}) = \Phi_{\mathcal{A}}(3,7) = 32$

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Table A.6: $\mathcal{A}_{3,8}$ with $\phi(\mathcal{A}_{3,8}) = \Phi_{\mathcal{A}}(3,8) = 44$

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Table A.7: $\mathcal{A}_{4,7}$ with $\phi(\mathcal{A}_{4,7}) = \Phi_{\mathcal{A}}(4,7) = 47$

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Table A.8: $\mathcal{A}_{4,8}$ with $\phi(\mathcal{A}_{4,8}) = \Phi_{\mathcal{A}}(4,8) = 84$

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A.3 List of arrangements with maximum average diameter and minimum number of external facets

The following tables list the arrangements with both $\Delta_{\mathcal{A}}(d, n)$ and $\Phi_{\mathcal{A}}(d, n)$ for $(d, n) = (3, 7)$, $(3, 8)$, $(4, 7)$, and $(4, 8)$.

Table A.9: $\mathcal{A}_{3,7}$ with $(\delta\mathcal{A}_{3,7}) = \Delta_{\mathcal{A}}(3,7) = \frac{9}{4}$ and $\phi(\mathcal{A}_{3,7}) = \Phi_{\mathcal{A}}(3,7) = 32$

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Table A.10: $\mathcal{A}_{3,8}$ with $(\delta\mathcal{A}_{3,8}) = \Delta_{\mathcal{A}}(3,8) = \frac{17}{7}$ and $\phi(\mathcal{A}_{3,8}) = \Phi_{\mathcal{A}}(3,8) = 44$

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Table A.11: $\mathcal{A}_{4,7}$ with $(\delta\mathcal{A}_{4,7}) = \Delta_{\mathcal{A}}(4,7) = \frac{11}{5}$ and $\phi(\mathcal{A}_{4,7}) = \Phi_{\mathcal{A}}(4,7) = 47$

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Table A.12: $\mathcal{A}_{4,8}$ with $\delta(\mathcal{A}_{4,8}) = \Delta_{\mathcal{A}}(4,8) = \frac{19}{7}$ and $\phi(\mathcal{A}_{4,8}) = \Phi_{\mathcal{A}}(4,8) = 84$

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