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Abstract

We study the value of component commonality in a periodic review assemble-to-order system, introduced by Akçay and Xu in 2004, where an independent base stock policy and a first-come-first-served allocation rule are used, and the base stock levels and the component allocation are optimized jointly. We show that lowering component commonality may yield a higher type-II service level. The lower degree of component commonality is achieved via separating inventories of the same component for different products. In their computational experiments, Akçay and Xu modified the right hand side of the inventory availability constraints by substituting linear functions for piece-wise linear ones. This modification may have a significant impact for low budget levels. The optimal solutions obtained via the original formulation; that is, without the modification, include zero base stock levels for some components and, thus, indicate a bias against component commonality. We substantiate this property via computational and theoretical approaches. We show that for low budget levels the use of separate inventories of the same component for different products could achieve a higher objective than with shared inventories. Finally, considering a simple assemble-to-order system consisting of one component shared by two products, we characterize the budget ranges such that the use of separate inventories is beneficial, as well as the budget ranges such that component commonality is beneficial.

1. Introduction

Given the pressure of high capital costs and the competitive environment in manufacturing, more and more manufacturers have adopted assemble-to-order (ATO) systems to increase product customization and reduce response time. The main difference between ATO and make-to-stock (MTS) approaches is that ATO eliminates the necessity for finished product inventories. When a customer order arrives, an ATO system

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satisfies the order by assembling the products from component inventories. Manufacturers will benefit from an ATO system if their product assembly times are negligible compared with their component replenishment lead times. While ATO systems provide numerous benefits, efficiently matching the demand and the supply for ATO systems is a challenging task. In particular, if the matching problem is not efficiently solved, those benefits may be offset, see Song and Zipkin [16]. In this paper, we focus on periodic review ATO systems with an independent base stock policy and a first-come-first-served allocation rule. We analyze from both theoretical and computational viewpoints the formulation of Akçay and Xu [2], which jointly optimizes the base stock levels and the component allocation. In particular, we discuss the impact of substituting linear inventory availability constraints for piece-wise linear ones in the formulation of Akçay and Xu [2], and the efficiency of component commonality for ATO systems.

1.1. Literature review

Component commonality is widely adopted and often preferred in ATO systems in order to offset the reduction of economies of scale when providing customized products. Extensive research has shown that component commonality is beneficial for single period models. Baker et al. [3] studied the effect of component commonality on optimal safety stock levels for an ATO system with two end-products and two components. They considered the problem of minimizing safety stock levels while satisfying a service level constraint under independent uniform demand distributions and showed that component commonality induced a reduction in the optimal safety stock levels. Gerchak et al. [7] extended this work by investigating whether the results hold for a system with an arbitrary number of products and a general joint demand distribution. Baker et al. [3] and Gerchak et al. [7] both assumed that the costs of the product-specific component and the common component are identical. However, it might be more realistic to assume that the common component is more expensive. Eynan and Rosenblatt [5] presented three models to compare and analyze the effects of increasing component commonality, demonstrating that some forms of commonality might not always be beneficial. They also provided conditions for which commonality should be either employed or avoided. Mirchandani and Mishra [13] compared a non-commonality model with two different commonality models – based on whether or not the products are prioritized – for a system with two products and independent uniform demand distributions. They derived theoretical conditions when component commonality is beneficial for this specific system.

While the above works assume a single period, the results could be quite different if a multi-period model is considered. Hillier [8] developed a simple multi-period model of a periodic review ATO system with zero lead times and uniformly distributed demands, and derived a closed-form solution for a cost minimization model with service level constraints. The results demonstrated that, for a multi-period model, the use of a common component is always beneficial if its price does not exceed that of the components it replaces. If the common component is more expensive than the ones it replaces, then in contrast to the single period case, it is almost never beneficial to use it. Hillier [9] further extended these results to systems with an arbitrary number of final products and components. Song and Zhao [15] considered a continuous-review ATO system with one common component, two end products, and Poisson distributed demands, and showed that, while component commonality is generally beneficial, its added value depends strongly on the component costs, lead times, and allocation rules.

Another stream of research studies the component commonality for systems with a given budget for all the components. Jönsson and Silver [11] analyzed the impact of component commonality for an ATO system with two end products and two components, with one being common to both products. Fong et al. [6] pursued the approach of Baker et al. [3] and provided analytical formulations for a commonality problem minimizing the expected shortage subject to a fixed budget constraint and assuming independent Erlang demand distributions. In particular, they observed that the relative reduction in the expected shortage can be substantial when the budget level is high relative to the demand requirements for the end products – even if the component is much more expensive. Other relevant works include Nonås [14], who formulated a two-stage stochastic program for an ATO system with three products and an arbitrary number of components, for which they developed a gradient-based search procedure to find the optimal inventory levels. Note that all these models assume a single period.

1.2. Akçay and Xu (2004) formulation

Following the model proposed by Akçay and Xu [2], we assume that :

(*i*) the review is periodic,
(*ii*) an independent base stock policy is used for each component,
(*iii*) the demands for products are satisfied using a first-come-first-served rule,
(*iv*) the random demands of products for each period could be correlated, while the demands of different periods are independent of each other,
(*v*) the replenishment lead time for each component is deterministic,
(*vi*) a reward is collected if the assembly is completed within the given time window.

In addition, the following sequence of events is assumed for each period:

(1) inventory position reviewed,
(2) new replenishment order of components placed,
(3) earlier replenishment order arrived,
(4) demand realized,
(5) component allocated and product assembled,
(6) associated rewards accounted for.

In this model, assembly takes zero time while component lead times are greater than zero. The model is based on a multi-matching approach proposed by Huang and de Kok [10] where multiple components are matched with multiple products to satisfy demands. In each period within the time window, rewards are collected by satisfying product demands. The base stocks of the ATO system are constrained by a pre-set overall budget. The approach is based on a two-stage decision model. The first stage consists of determining a base stock level for each component, and the second stage consists of determining products that need to be assembled in each period with respect to some constraints reflecting the inventory availability. The first stage decisions are made before the second stage decisions following a two-stage stochastic programming framework, see Birge and Louveaux [4]. The objective of the approach is to maximize the expected total reward collected from the products assembled within given time windows.

The notation used for variables is as follows:

n	:	number of components
m	:	number of products
i	:	index of component $i = 1, \ldots, n$
j	:	index of product $j = 1, \ldots, m$
S_i	:	base stock level of component $i = 1, \ldots, n$
c_i	:	unit base stock level cost of component $i = 1, \ldots, n$
L_i	:	lead time of component $i = 1, \ldots, n$
L	:	maximum lead time among all components; that is., $L = \max_{i=1}^{n} L_i$
w_j	:	time window of product j
k	:	index of period k corresponding to the duration $[k, k+1)$; $k = 0$ implies the current period; negative values of k imply previous periods
$x_{j,k}$	:	number of product j assembled in period k
$r_{j,k}$	:	reward for satisfying the demand for product j in period k
$a_{i,j}$	:	number of component i used to assemble one unit of product j; that is, the bill of materials (BOM)
B	:	the budget, i.e., $\sum_{i=1}^{n} c_i \times S_i \leq B$
$P_{j,k}$	:	demand of product j at period k
P_j	:	demand of product j at the current period, i.e., $P_{j,0}$
$D_{i,k}$	:	demand of component i at period k, i.e., $D_{i,k} = \sum_{j=1}^{n} a_{i,j} P_{j,k}$
M	:	number of independent samples
N	:	number of realizations in one sample
l	:	index of sample $l = 1, \ldots, M$
h	:	index of realization $h = 1, \ldots, N$
x^+	:	the positive part of x, that is, $x^+ = (\lvert x \rvert + x)/2$

The second stage corresponds to the allocation problem $\big(Alloc(S, \xi)\big)$, where $S = (S_i)$

is the vector representing base stock levels, $\xi = \{P_{j,k} | j = 1, \ldots, m; k = 0, -1, \ldots, -L\}$ is the vector representing random demands, and $O_{i,k}$ is the number of component i available at period k. Note that $O_{i,k} = (S_i - D_i^{L_i - k})^+$ for $0 \leq k \leq L_i$ where $D_i^{L_i-k} = \sum_{s=0}^{L_i-k} D_{i,-s}$, and $O_{i,k} = D_{i,0}$ for $L_i + 1 \leq k \leq L + 1$ following the base stock policy and a first-come-first-served rule, see Huang and de Kok [10].

$$\begin{aligned} \max \quad & \sum_{j=1}^{m} \sum_{k=0}^{w_j} r_{j,k} \times x_{j,k} && \big(Alloc(S,\xi)\big) \\ & \sum_{k=0}^{L+1} x_{j,k} = P_j && j = 1, \ldots, m \\ \sum_{\mu=0}^{k} \sum_{j=1}^{m} & a_{i,j} \times x_{j,\mu} \leq O_{i,k} && i = 1, \ldots, n, \quad k = 0, \ldots, L+1 \\ & x_{j,k} \in \mathbb{Z}_+ && j = 1, \ldots, m, \quad k = 0, \ldots, L+1 \end{aligned}$$

The first set of constraints guarantees that assembly will satisfy customer demand. The second set of constraints – called inventory availability constraints – guarantees that assembly could only happen when there are enough component inventories. While an optimal allocation can be computed for a given base stock level S and demand ξ, we still need to determine the optimal base stock levels. Thus, we use the two-stage stochastic integer program $\big(Joint(B)\big)$ where the first stage determines the base stock levels and the the second stage maximizes the expectation of the component allocations:

$$\begin{aligned} \max \quad & \mathbf{E}[Alloc(S,\xi)] && \big(Joint(B)\big) \\ & \sum_{i=1}^{n} c_i \times S_i \leq B \\ & S_i \in \mathbb{Z}_+ && i = 1, \ldots, n \end{aligned}$$

We recall in Section 1.3 the sample average approximation method used to solve $\big(Joint(B)\big)$.

1.3. Sample average approximation method

The sample average approximation (SAA) method, see Kleywegt et al. [12], consists of the following steps:

(i) generate M independent samples for $l = 1, \ldots, M$ with N realizations for each sample. The vector $\xi_l^N = (\xi(\omega_l^1), \xi(\omega_l^2), \ldots, \xi(\omega_l^N))$ represents the N realizations of the l-th sample,

(ii) solve the optimization problem ($INLP$) for each sample, which is the associated deterministic version of $\big(Joint(B)\big)$. where the objective function is set to $\frac{1}{N}\sum_{h=1}^{N} Alloc(S,\xi(\omega_l^h))$ as described below. Note that ($INLP$) is non-linear not only due to the integrality constraints but also due to the right hand side of the inventory availability constraints. Let $\hat{S}_l$ denote the optimal base stock levels for ($INLP$) and $\hat{G}(\hat{S}_l)$ denote its optimal objective value.

$$
\begin{aligned}
\max \quad & \frac{1}{N}\sum_{h=1}^{N}\sum_{j=1}^{m}\sum_{k=0}^{w_j} r_{j,k}\times x_{j,k}^h && && \big(INLP\big)\\
& \sum_{k=0}^{L+1} x_{j,k}^h = P_j^h && j=1,\ldots,m, \quad h=1,\ldots,N\\
& \sum_{\mu=0}^{k}\sum_{j=1}^{m} a_{i,j}\times x_{j,\mu}^h \le O_{i,k}^h && i=1,\ldots,n, \quad k=0,\ldots,L+1, \quad h=1,\ldots,N\\
& \sum_{i=1}^{n} c_i\times S_i \le B\\
& S_i\in\mathbb{Z}_+ && i=1,\ldots,n\\
& x_{j,k}^h\in\mathbb{Z}_+ && j=1,\ldots,m, \quad k=0,\ldots,L+1, \quad h=1,\ldots,N
\end{aligned}
$$

(iii) generate a different sample $\xi^{N'}$ with $N'\gg N$ realizations and compare the performance among all the base stock vectors $\hat{S}_l$ solved in (ii) by solving $\big(Alloc(S,\xi^{N'})\big)$ with $S=\hat{S}_l$. Let $\bar{G}(\hat{S}_l)$ be the new optimal objective value.

(iv) select the optimal base stock vector $\hat{S}^*$ achieving the best performance among all the base stock vectors; that is, $\hat{S}^*=\text{argmax}\{\bar{G}(\hat{S}_l): l=1,\ldots,M\}$.

Let $\hat{G}_M=\frac{1}{M}\sum_{l=1}^{M}\hat{G}(\hat{S}_l)$, $\bar{G}_{N'}=\bar{G}(\hat{S}^*)$, and G^* be the optimal objective value of $\big(Joint(B)\big)$. Since $\bar{G}_{N'}\le G^*\le\hat{G}_M$ under certain conditions for N,M,N', see Birge and Louveaux [4], $\bar{G}_{N'}$ and $\hat{G}_M$ are, respectively, a lower and an upper bound for G^*. For more details concerning the statistical testing of optimality for the SAA method, and the selection of N, M, and N', see Kleywegt et al. [12].

2. Impact of modifying the inventory availability constraints

While the ($INLP$) formulation uses a plus sign in the right hand side of the inventory availability constraints, $(S_i-D_i^{L_i-k})^+$ is substituted by $(S_i-D_i^{L_i-k})$ in the computational experiments performed by Akçay [1]. The obtained new formulation (ILP) allows faster computation. Note that the feasible region of (ILP) is a subset of the feasible region of (*INLP*). In addition, while relaxing the integrality constraints on the variables would make (*ILP*) convex, (*INLP*) would remain non-convex due to the $(S_i - D_i^{L_i-k})^+$ term in the right hand side of the inventory availability constraints. Note that substituting $(S_i - D_i^{L_i-k})$ for $(S_i - D_i^{L_i-k})^+$ may lead to infeasibility. This issue can be addressed by filtering out samples leading to infeasibility and by assuming sufficiently large budget level; that is, by allowing large base stock levels. We argue that substituting $(S_i - D_i^{L_i-k})$ for $(S_i - D_i^{L_i-k})^+$ might yield an intractable sample generation process for the SAA approach for low budget levels.

2.1. Impact of modifying the inventory availability constraints on the sample generation

Generating enough samples such that the associated (*ILP*) formulation is feasible could be highly challenging for low budget levels. Note that under the extreme case of setting the budget to zero, the only sample yielding a feasible formulation is the trivial zero sample. Disregarding infeasible ones, we generate samples for (*ILP*) until the required number of samples, or a pre-set number of feasibility tests, is reached. For a given budget, the feasibility check is done by comparing with a computed minimum budget having a feasible solution. The computed minimum budget is determined from the (*ILP*) minimum base stock levels using Algorithm 1 described below. The non-negativity of the left hand side of the inventory availability constrains implies $(S_i - D_i^{L_i-k}) \geq 0$. Note that while we can generate enough feasible samples for (*ILP*), the mean and variance of generated sample are possibly impacted and, thus, the SAA method.

Algorithm 1 Computing minimum feasible budget

```
Initialize maxS ← zeros(n)
for any realization h do
    for for any component i do
        if D_i^{L_i} > maxS(i) then
            maxS(i) ← D_i^{L_i}
        end if
    end for
end for
B = Σ_{i=1}^{n} c_i × maxS(i)
```

2.2. Impact of modifying the inventory availability constraints on the SAA method

Following the notation and discussion of Section 1.3, let $\bar{G}^{\bullet}_{N'}$, $G^{*}_{\bullet}$, and $\hat{G}^{\bullet}_{M}$ denote respectively the (*ILP*) lower bound, optimal value, and upper bound. Since $x \leq x^+$, any feasible solution of (*ILP*) is a feasible solution of (*INLP*). In addition, this inclusion is typically strict as one can set some base stocks to zero to build a solution feasible for

(*INLP*) but infeasible for (*ILP*). To ensure a fair comparison, we only consider samples yielding feasible (*ILP*) and (*INLP*) formulations. Since, for a given sample, the optimal objective value for (*INLP*) is at least the one for (*ILP*), we have $G^{\bullet}_M \leq \hat{G}_M$.

2.3. Computational results for Zhang's system (1997)

We consider an ATO system proposed by Zhang [17] with four products and five components as described in Table 2. The computational results are presented in Table 3 where N/A corresponds to budgets for which not enough sample yielding a feasible formulation can be generated, and LB and UB denote, respectively, the lower and upper bounds for the (*ILP*) and (*INLP*) formulations. The parameters for the SAA method are set to: $N = 25, N' = 100$ and $M = 5000$. If a million samples are not enough to yield 100 feasible (*INLP*) formations, the process stops and outputs N/A.

						Component				
					i	1	2	3	4	5
					c_i	2	3	6	4	1
		Product			L_i	3	1	2	4	4
j	*Mean*	*StdDev*	r_j	w_j		Bill of Materials				
1	100	25	1	0		1	2	1	0	0
2	150	30	1	0		1	1	1	0	0
3	50	15	1	0		0	1	1	1	0
4	30	11	1	0		0	0	0	1	1

Table 2: Settings for Zhang's system

Budget	**(*ILP*)-LB**	**(*ILP*)-UB**	**(*INLP*)-LB**	**(*INLP*)-UB**
2000	*N/A*	*N/A*	9.08	9.11
3000	*N/A*	*N/A*	9.08	9.12
4000	*N/A*	*N/A*	9.46	9.88
5000	*N/A*	*N/A*	21.59	22.98
6000	*N/A*	*N/A*	46.47	47.83
7000	*N/A*	*N/A*	65.78	66.49
8000	68.58	70.54	74.88	75.01
9000	87.85	88.85	89.07	90.02
10000	97.76	98.12	98.20	98.34

Table 3: Type-II service levels for Zhang's system with different budgets

3. Component commonality for specific ATO systems

3.1. Component commonality for Zhang's system

The computational experiments performed for Zhang's system with (*INLP*) formulation show that, for some low level budgets, the optimal base stock levels of some components are set to zero, see Table 4. This computation indicates a bias against component commonality and suggests that dedicating the components to different products may yield a higher objective value. For example, for a budget 2000, the inventory levels for C_1, C_2 and C_3 are set to zero implying that an optimal solution only considers assembling product 4. Similarly, for a budget between 5000 and 8000, the optimal base stock levels for components C_4 and C_5 are set to zero and thus implies products 3 and 4 are ignored. Note that while all products are eventually assembled within $L+1$ periods, the rewards are collected only within the pre-set time windows.

Budget	C_1	C_2	C_3	C_4	C_5	**LB**	**UB**
2000	0	0	0	428	199	9.08	9.11
3000	0	0	162	413	376	9.08	9.12
4000	0	325	249	339	175	9.46	9.88
5000	613	492	383	0	0	21.59	22.98
6000	699	598	468	0	0	46.47	47.83
7000	782	722	545	0	0	65.78	66.49
8000	865	846	622	0	0	74.88	75.01
9000	793	779	595	339	151	89.07	90.02
10000	855	876	665	377	163	98.20	98.34

Table 4: Optimal base stock levels and Type-II service levels for Zhang's system given different budgets

We propose a model separating component inventories with respect to the different products; that is, each product is served by dedicated components. We consider a modified bill of material (BOM) for Zhang's system as described in Table 5. In the first row, the subscript is the component index in the original BOM, and the superscript is the index of the product served bye the component. The components with the same subscript have the same cost and lead time. Computational experiments, presented in Table 6, are performed to compare Zhang's system with maximum component commonality, denoted as (*INLP*), and Zhang's system with no component commonality, denoted as ($INLP_{\Delta}$). Table 6 indicates that the ($INLP_{\Delta}$) model outperforms the original (*INLP*) model for a budget no greater than 8000. While the gap decreases with the increase of the budget, it is significant for a low to medium budget.

	C_1^1	C_2^1	C_3^1	C_1^2	C_2^2	C_3^2	C_2^3	C_3^3	C_4^3	C_4^4	C_5^4
P_1	1	2	1	0	0	0	0	0	0	0	0
P_2	0	0	0	1	1	1	0	0	0	0	0
P_3	0	0	0	0	0	0	1	1	1	0	0
P_4	0	0	0	0	0	0	0	0	0	1	1

Table 5: Bill of materials for Zhang's system without component commonality

Budget	**(*INLP*)-LB**	**(*INLP*)-UB**	**($INLP_\Delta$)-LB**	**($INLP_\Delta$)-UB**
2000	29.98	30.08	47.07	47.51
3000	29.98	30.08	84.53	87.78
4000	31.20	32.61	143.66	144.22
5000	71.26	75.83	177.27	177.10
6000	153.35	157.83	192.86	196.02
7000	217.08	219.42	226.80	230.28
8000	247.11	247.54	263.13	264.80
9000	293.92	297.08	278.47	281.45
10000	324.06	324.52	310.37	312.46

Table 6: Objective value estimation for (*INLP*) and ($INLP_\Delta$)

3.2. Component commonality for Λ*-system*

While in Section 3.1, the gap between the ($INLP_\Delta$) and (*INLP*) models is substantiated computationally, we provide a theoretical analysis for a simpler system, denoted Λ-system, consisting of one component shared by two products. See Table 7 for a description of the original Λ-system and of our modified model, denoted Λ_Δ-system, that removes component commonality.

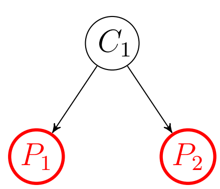
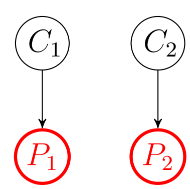
Λ-system	Λ_Δ-system
	

Table 7: Bill of materials for Λ- and Λ_Δ-systems

	Mean	*StdDev*	r_j
P_1	100	25	1
P_2	150	30	1

Table 8: Parameters for Λ- and Λ_Δ-systems

To simplify the analysis, the component costs, product rewards, and product time windows are all set to 1. The corresponding SAA formulations ($INLP^N$) and ($INLP^N_\Delta$) are as follows:

$$\begin{aligned} \max \quad & \frac{1}{N}\sum_{h=1}^{N}(x_1^h + x_2^h) && \quad (INLP^N)\\ x_1^h + x_2^h \leq (B - D_1^h - D_2^h)^+ & && h = 1,\dots,N\\ x_1^h \leq P_1^h, \quad x_2^h \leq P_2^h & && h = 1,\dots,N\\ x_1^h, x_2^h \in \mathbb{Z}_+ & && h = 1,\dots,N \end{aligned}$$

$$\begin{aligned} \max \quad & \frac{1}{N}\sum_{h=1}^{N}(x_1^h + x_2^h) && \quad (INLP^N_\Delta)\\ x_1^h \leq (B_1 - D_1^h)^+ & && h = 1,\dots,N\\ x_2^h \leq (B_2 - D_2^h)^+ & && h = 1,\dots,N\\ x_1^h \leq P_1^h, \quad x_2^h \leq P_2^h & && h = 1,\dots,N\\ B_1 + B_2 = B & && \\ x_1^h, x_2^h \in \mathbb{Z}_+ & && h = 1,\dots,N\\ B_1, B_2 \in \mathbb{R}_+. & && \end{aligned}$$

Theorem 3.1 characterizes the budget ranges such that component commonality is beneficial for Λ-system over Λ_Δ-system. For example, the "$<$" sign means that common commonality is non-beneficial for a budget ranging from B_{min} to B^+_{min} as specified in Theorem 3.1. The proof of Theorem 3.1 is given in Section 4.

Theorem 3.1. *Given a budget B, let $f^*(B)$ and $f^*_\Delta(B)$ be the optimal objective values of ($INLP^N$) and ($INLP^N_\Delta$). Considering the cases $N = 1, 2$ and N_0, the sign of $f^*(B)$ - $f^*_\Delta(B)$ is given in Table 9 where $B_{min} = \min_{i=1}^2\{\min_{h=1}^N\{D_i^h\}\}$, $B^+_{min} = \min_{h=1}^N\{D_1^h + D_2^h\}$, $B^+_{max} = \max_{h=1}^N\{D_1^h + D_2^h\}$, and $B^\Sigma_{max} = \sum_{i=1}^2 \max_{h=1}^N\{D_i^h + P_i^h\}$.*

B \ N	$[0, B_{min}]$	$(B_{min}, B^+_{min}]$	$(B^+_{min}, B^+_{max}]$	$(B^+_{max}, B^{\Sigma}_{max}]$	$(B^{\Sigma}_{max}, +\infty)$
1	$=$	$<$	$<$	$\leq$	$=$
2	$=$	$<$	$\leq$	$\leq$ or $>$	$=$
N_0	$=$	$<$	$\leq$ or $>$	$\leq$ or $>$	$=$

Table 9: Λ_{Δ}-system where a positive sign corresponds to common commonality being beneficial

4. Proof of Theorem 3.1

4.1. Case $N = 1$

We first consider the case $N = 1$; that is for one realization in the SAA method. The associated formulations $(INLP^1)$ and $(INLP^1_{\Delta})$ correspond to a deterministic demand where P^1_1 and P^1_2 represent the demands in the current period for, respectively, product 1 and 2, and D^1_1 and D^1_2 represent the overall demands from all previous periods. The budget level B is given and since the cost of the component is set to one, the budget level is equivalent to the base stock level.

$$\begin{aligned} \max \quad & x^1_1 + x^1_2 \\ x^1_1 + x^1_2 &\leq (B - D^1_1 - D^1_2)^+ \\ x^1_1 \leq P^1_1, &\quad x^1_2 \leq P^1_2 \\ x^1_1, x^1_2 &\in \mathbb{Z}_+ \end{aligned} \tag{$INLP^1$}$$

$$\begin{aligned} \max \quad & x^1_1 + x^1_2 \\ x^1_1 &\leq (B_1 - D^1_1)^+ \\ x^1_2 &\leq (B_2 - D^1_2)^+ \\ x^1_1 \leq P^1_1, &\quad x^1_2 \leq P^1_2 \\ B_1 + B_2 &= B \\ x^1_1, x^1_2 &\in \mathbb{Z}_+ \\ B_1, B_2 &\in \mathbb{R}_+ \end{aligned} \tag{$INLP^1_{\Delta}$}$$

Property 1. *Given a budget B, let $f^*(B)$ and $f^*_{\Delta}(B)$ be the optimal objective values of $\left(INLP^1\right)$ and $\left(INLP^1_{\Delta}\right)$. Both $f^*(B)$ and $f^*_{\Delta}(B)$ are monotonically non-decreasing with B and $f^*(B) \leq f^*_{\Delta}(B)$.*

Proof. Since the feasible region of $\left(INLP^1\right)$ for a given B is a subset of the feasible region of $\left(INLP^1\right)$ f for $B' \geq B$, $f^*(B)$ is non-decreasing with B increasing. The same holds for $f^*_{\Delta}(B)$. We then prove that $f^*(B){\leq}f^*_{\Delta}(B)$ by showing that an optimal solution for $\left(INLP^1\right)$ yields a feasible solution for $\left(INLP^1_{\Delta}\right)$. Assume first that an optimal solution for $\left(INLP^1\right)$ satisfies $(x^1_1)^* = 0$. Then, the solution $\hat{x}^1_1 = (x^1_1)^* = 0$,

$\hat{x}_2^1 = (x_2^1)^*$, $B_1 = 0$, $B_2 = B$ is feasible for $\left(INLP^1_\Delta\right)$ as $\hat{x}_2^1 \le (B - D_2^1)^+$ holds since $\hat{x}_2^1 = (x_2^1)^* \le (B - D_1^1 - D_2^1)^+ \le (B - D_2^1)^+$. Assume then that an optimal solution for $\left(INLP^1\right)$ satisfies $(x_1^1)^* > 0$. Then, the solution $\hat{x}_1^1 = (x_1^1)^*$, $\hat{x}_2^1 = (x_2^1)^*$, $B_1 = (x_1^1)^* + D_1^1$, $B_2 = B - (x_1^1)^* - D_1^1$ is feasible for $\left(INLP^1_\Delta\right)$ as $\hat{x}_2^1 = (x_2^1)^* \le (B - (x_1^1)^* - D_1^1 - D_2^1)^+$ holds since $(x_1^1)^* > 0$ implies $B > D_1^1 + D_2^1$; that is : $B - D_1^1 - D_2^1 \ge (x_1^1)^* + (x_2^1)^*$ by the first constraint of $\left(INLP^1\right)$. □

Property 2 refines the inequality $f^*(B) \le f^*_\Delta(B)$ for $N = 1$ by providing budget ranges for which the inequality is strict or holds with equality.

Property 2. *Given a budget B, let $f^*(B)$ and $f^*_\Delta(B)$ be the optimal objective values of $\left(INLP^1\right)$ and $\left(INLP^1_\Delta\right)$. We have:*

$f^*(B) = f^*_\Delta(B)$ if $B \le B_{min}$ or $B \ge D_1^1 + D_2^1 + \max\{P_1^1, P_2^1\}$, and
$f^*(B) < f^*_\Delta(B)$ if $B_{min} < B < D_1^1 + D_2^1 + \max\{P_1^1, P_2^1\}$.

Proof. Consider first the case $B \le B_{min} = \min(D_1^1, D_2^1)$, then $(x_1^1)^* = (x_2^1)^* = (\hat{x}_1^1)^* = (\hat{x}_2^1)^* = 0$, and thus $f^*(B) = f^*_\Delta(B) = 0$. Consider then the case $B \ge D_1^1 + D_2^1 + \max\{P_1^1, P_2^1\}$. Adding the last two constraints of $\left(INLP^1_\Delta\right)$ yields that $P_1^1 + P_2^1$ is an upper bound; that is, $f^*(B) \le f^*_\Delta(B) \le P_1^1 + P_2^1$. Without loss of generality, we assume $P_1^1 > P_2^1$ and consider two sub-cases. Sub-case $B \ge D_1^1 + D_2^1 + P_1^1 + P_2^1$: then the solution $(x_1^1)^* = P_1^1$ and $(x_2^1)^* = P_2^1$ is feasible for $\left(INLP^1\right)$ and, thus, $P_1^1 + P_2^1 \le f^*(B) \le f^*_\Delta(B) \le P_1^1 + P_2^1$ which implies $f^*(B) = f^*_\Delta(B)$. Sub-case $D_1^1 + D_2^1 + \max\{P_1^1, P_2^1\} \le B < D_1^1 + D_2^1 + P_1^1 + P_2^1$: then an optimal solution for $\left(INLP^1\right)$ satisfies $(x_1^1)^* = P_1^1$ and $(x_2^1)^* = B - D_1^1 - D_2^1 - P_1^1$. Furthermore, for $\left(INLP^1_\Delta\right)$, if $B_1 - D_1^1 < 0$ then $x_1^1 = 0$ and $x_2^1 \le P_2^1 < P_1^1 < f^*(B)$which is not an optimal solution, therefore we can assume that $B_1 - D_1^1 \ge 0$. In addition, if $B_2 - D_2^1 < 0$ then $x_2^1 = 0$ and $x_1^1 \le P_1^1$ which can not yield a strictly larger objective value. Thus we can assume $B_1 - D_1^1 \ge 0$ and $B_2 - D_2^1 \ge 0$. Adding the first two constraints shows that $f^*_\Delta(B) \le B - D_1^1 - D_2^1$, and thus a strictly larger objective value can not be achieve; that is $f^*(B) = f^*_\Delta(B)$. Finally, consider the case $\min(D_1^1, D_2^1) < B < D_1^1 + D_2^1 + \max\{P_1^1, P_2^1\}$. We consider 2 sub-cases. Sub-case $\min(D_1^1, D_2^1) < B \le D_1^1 + D_2^1$: then $f^*(B) = 0$ while $B_1^* = B$ and $B_2^* = 0$ yields a feasible solution for $\left(INLP^1_\Delta\right)$ which a strictly positive objective value and, thus, $f^*_\Delta(B) > f^*(B)$. Sub-case $D_1^1 + D_2^1 < B < D_1^1 + D_2^1 + \max\{P_1^1, P_2^1\}$ and, without loss of generality, $P_1^1 > P_2^1$: then $f^*(B) \le B - D_1^1 - D_2^1 < P_1^1$ by the first constraint of $\left(INLP^1\right)$. On the other hand, setting $B_1^* = B$, $B_2^* = 0$ and $\hat{x}_1^1 = \min\{B - D_1^1, P_1^1\}$ yields a feasible solution for $\left(INLP^1_\Delta\right)$ with an objective value of at least P_1^1; that is, $f^*_\Delta(B) \ge P_1^1 > f^*(B)$. □

4.2. Case $N = 2$

We consider the case $N = 2$; that is the simplest random demand with only two realizations. We assume that both realizations have probability 0.5 and omit this constant term in the objectives for clarity. In the associated formulations ($INLP^2$) and ($INLP^2_\Delta$) below, superscripts are use to distinguish different realizations. For example, x_1^2, D_1^2, and P_1^2 refer to the second realization.

$$\begin{aligned} \max \quad & x_1^1 + x_2^1 + x_1^2 + x_2^2 && \left(INLP^2\right) \\ x_1^1 + x_2^1 & \leq (B - D_1^1 - D_2^1)^+ \\ x_1^2 + x_2^2 & \leq (B - D_1^2 - D_2^2)^+ \\ x_1^1 \leq P_1^1, & \quad x_2^1 \leq P_2^1 \\ x_1^2 \leq P_1^2, & \quad x_2^2 \leq P_2^2 \\ x_1^1, x_2^1, x_1^2, x_2^2 & \in \mathbb{Z}_+ \end{aligned}$$

$$\begin{aligned} \max \quad & x_1^1 + x_2^1 + x_1^2 + x_2^2 && \left(INLP^2_\Delta\right) \\ x_1^1 & \leq (B_1 - D_1^1)^+ \\ x_2^1 & \leq (B_2 - D_2^1)^+ \\ x_1^2 & \leq (B_1 - D_1^2)^+ \\ x_2^2 & \leq (B_2 - D_2^2)^+ \\ x_1^1 \leq P_1^1, & \quad x_2^1 \leq P_2^1 \\ x_1^2 \leq P_1^2, & \quad x_2^2 \leq P_2^2 \\ B_1 + B_2 & = B \\ x_1^1, x_2^1, x_1^2, x_2^2 & \in \mathbb{Z}_+ \\ B_1, B_2 & \in \mathbb{R}_+ \end{aligned}$$

As the number of cases to consider in order to provide an analogue of Property 2 essentially increases exponentially with the number of realizations, comparing ($INLP^2$) and ($INLP^2_\Delta$) can be quite tedious. Thus, Property 3. focuses on the following 3 scenarios : (*i*) the demands are large for both realizations, (*ii*) the demands are large for one realization and small for the other, and (*iii*) the demands are small for both realizations.

Property 3. *Given a budget B, let $f^*(B)$ and $f^*_\Delta(B)$ be the optimal objective values of $\left(INLP^2\right)$ and $\left(INLP^2_\Delta\right)$ We have:*

$f^*(B) < f^*_\Delta(B)$ if $B_{min} < B \leq B^+_{min}$,
$f^*(B) \leq f^*_\Delta(B)$ if $B^+_{min} < B \leq B^+_{max}$, and
$f^*(B) = f^*_\Delta(B)$ if $0 \leq B \leq B_{min}$ or $B \geq \max\{D_1^1 + P_1^1, D_1^2 + P_1^2\} + \max\{D_2^1 + P_2^1, D_2^2 + P_2^2\}$.

Proof. Consider first the case $B \leq B_{min}$, then $(x_1^1, x_1^2, x_2^1, x_2^2)$ must be set to $(0, 0, 0, 0)$ to obtain a feasible solution for $\left(INLP^2\right)$ and $\left(INLP^2_\Delta\right)$ and, thus, $f^*(B) = f^*_\Delta(B) = 0$.

Consider the case $B \geq \max\{D_1^1+P_1^1, D_1^2+P_1^2\}+\max\{D_2^1+P_2^1, D_2^2+P_2^2\}$. First note that $P_1^1+P_2^1+P_1^2+P_2^2$ is an upper bound both $f^*(B)$ and $f^*_\Delta(B)$ as implied by adding the last 4 constraints. Then, as $x_i^h = P_i^h$ is a feasible solution for $\left(INLP^2\right)$, $f^*(B) = P_1^1+P_2^1+P_1^2+P_2^2$. Similarly, $x_i^h = P_i^h$, $B_1 = \max\{D_1^1+P_1^1, D_1^2+P_1^2\}$ and $B_2 = B - B_1$ a feasible solution for $\left(INLP^2_\Delta\right)$ and the corresponding objective is also $P_1^1+P_2^1+P_1^2+P_2^2$; that is, $f^*_\Delta(B)= f^*(B)$. Consider the case $B \leq B^+_{min} = \min\{D_1^1+D_2^1, D_1^2+D_2^2\}$, then while $f^*(B) = 0$, setting $B_1^* = B$ and $B_2^* = 0$ yields a feasible solution for $\left(INLP^2_\Delta\right)$ with a strictly positive objective value; that is, $f^*(B) < f^*_\Delta(B)$. Consider the case $B^+_{min} < B \leq B^+_{max}$, and assume without loss of generality that $D_1^2+D_2^2 > D_1^1+D_2^1$. Since $B \leq D_1^2+D_2^2$, the second constraints of $\left(INLP^2\right)$ is $x_1^2+x_2^2 \leq 0$; that is $x_1^2 = x_2^2 = 0$. In other words, we can restrict to $(x_1^1, x_2^1, 0, 0)$ feasible solutions and use Property 2 to derive $f^*(B) \leq f^*_\Delta(B)$. □

4.3. Case $N = N_0$

Similarly to Section 4.2, we assume for $N = N_0$, that the N_0 realizations have probability $1/N_0$ and omit this constant term in the objectives for clarity. In the associated formulations $(INLP^{N_0})$ and $(INLP^{N_0}_\Delta$ below, superscripts are use to distinguish different realizations. For example, $x_1^h, x_2^h, D_1^h, D_2^h, P_1^h$, and P_1^h refer to the h-th realization.

$$\begin{aligned}
\max \quad & \sum_{h=1}^{N_0}(x_1^h + x_2^h) && \left(INLP^{N_0}\right)\\
x_1^h + x_2^h &\leq (B - D_1^h - D_2^h)^+ && h = 1, \ldots, N_0\\
x_1^h \leq P_1^h, \quad & x_2^h \leq P_2^h && h = 1, \ldots, N_0\\
x_1^h, x_2^h &\in \mathbb{Z}_+ && h = 1, \ldots, N_0
\end{aligned}$$

$$\begin{aligned}
\max \quad & \sum_{h=1}^{N_0}(x_1^h + x_2^h) && \left(INLP^{N_0}_\Delta\right)\\
x_1^h &\leq (B_1 - D_1^h)^+ && h = 1, \ldots, N_0\\
x_2^h &\leq (B_2 - D_2^h)^+ && h = 1, \ldots, N_0\\
x_1^h \leq P_1^h, \quad & x_2^h \leq P_2^h && h = 1, \ldots, N_0\\
B_1 + B_2 &= B\\
x_1^h, x_2^h &\in \mathbb{Z}_+ && h = 1, \ldots, N_0\\
B_1, B_2 &\in \mathbb{R}_+.
\end{aligned}$$

Similarly to Section 4.2, the number of cases being essentially intractable, Property 4. focuses on the following 2 scenarios : (i) the demands are large for all N_0 realizations, and (ii) the demands are small for all N_0 realizations.

Property 4. *Given a budget B, let $f^*(B)$ and $f^*_\Delta(B)$ be the optimal objective values of $\left(INLP^{N_0}\right)$ and $\left(INLP^{N_0}_\Delta\right)$ We have:*

$f^*(B)= f^*_\Delta(B)$ *if* $0 \le B \le B_{min}$ *or* $B \ge B^\Sigma_{max}$, *and*
$f^*(B)< f^*_\Delta(B)$ *if* $B_{min} < B \le B^+_{min}$.

Proof. Similarly to Property 3, for $B \le B_{min}$, x^h_i must be set to 0 for $h = 1, \dots, N_0$ and $i = 1$ and 2 to obtain a feasible solution for $\left(INLP^{N_0}\right)$ and $\left(INLP^{N_0}_\Delta\right)$ and, thus, $f^*(B) = f^*_\Delta(B) = 0$. Consider then, Since $B \ge B^\Sigma_{max} = \sum_{i=1}^2 \max_{h=1}^{N_0}\{D^h_i + P^h_i\}$. First note that $\sum_{i=1}^2 \sum_{h=1}^{N_0}\{P^h_i\}$ is an upper bound both $f^*(B)$ and $f^*_\Delta(B)$ as implied by adding the last $2N_0$ constraints. Then, as $x^h_i = P^h_i$ is a feasible solution for $\left(INLP^{N_0}\right)$, $f^*(B) = \sum_{i=1}^2 \sum_{h=1}^{N_0}\{P^h_i\}$. Similarly, $x^h_i = P^h_i$, $B_1 = \max_{h=1}^{N_0}\{D^h_1 + P^h_1\}$ and $B_2 = B - B_1$ a feasible solution for $\left(INLP^{N_0}_\Delta\right)$ and the corresponding objective is also $\sum_{i=1}^2 \sum_{h=1}^{N_0}\{P^h_i\}$; that is, $f^*_\Delta(B) = f^*(B)$.

Consider the case $B \le B^+_{min} = \min_{h=1}^{N_0}\{D^h_1 + D^h_2\}$, then while $f^*(B) = 0$, setting $B^*_1 = B$ and $B^*_2 = 0$ yields a feasible solution for $\left(INLP^{N_0}_\Delta\right)$ with a strictly positive objective value; that is, $f^*(B) < f^*_\Delta(B)$. □

5. Conclusions and future work

We highlighted the critical role played by the piece-wise linear inventory availability constraints and the associated feasibility issue and challenges for sample generation. The computational results estimate the impact resulting from substituting linear functions for piece-wise linear ones: While the impact decreases when the budget increases, it remains significant for low to medium level budgets. In addition, the benefits of component commonality are analyzed from theoretical and computational aspects and illustrated for specific ATO systems. We introduce a simple inventory control method applicable in practice where a more flexible design of products and componentsallows us to exploit the different degrees of component commonality according to the budget. Future works include an enhanced analysis of the sample generation process for (ILP) and a tighter estimate of the gap between the optimal objective values of (ILP) and ($INLP$). Further flexibility for the proposed inventory control method might be achieved via component commonality for subset of components and products.

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