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Abstract

A lattice (d,k)-polytope is the convex hull of a set of points in dimension d whose coordinates are integers between 0 and k. Let $\delta(d,k)$ be the largest diameter over all lattice (d,k)-polytopes. We develop a computational framework to determine $\delta(d,k)$ for small instances. We show that $\delta(3,4)=7$ and $\delta(3,5)=9$; that is, we verify for $(d,k)=(3,4)$ and $(3,5)$ the conjecture whereby $\delta(d,k)$ is at most $\lfloor (k+1)d/2 \rfloor$ and is achieved, up to translation, by a Minkowski sum of lattice vectors.

Keywords: Lattice polytopes, edge-graph diameter, enumeration algorithm

1 Introduction

Finding a good bound on the maximal edge-diameter of a polytope in terms of its dimension and the number of its facets is not only a natural question of discrete geometry, but also historically closely connected with the theory of the simplex method, as the diameter is a lower bound for the number of pivots required in the worst case. Considering bounded polytopes whose vertices are rational-valued, we investigate a similar question where the number of facets is replaced by the grid embedding size.

The convex hull of integer-valued points is called a lattice polytope and, if all the vertices are drawn from $\{0,1,\ldots,k\}^d$, it is referred to as a lattice (d,k)-polytope. Let $\delta(d,k)$ be the largest edge-diameter over all lattice (d,k)-polytopes. Naddef [7] showed in 1989 that $\delta(d,1)=d$, Kleinschmidt and Onn [6] generalized this result in 1992 showing that $\delta(d,k)\leq kd$. In 2016, Del Pia and Michini [3] strengthened the upper bound to $\delta(d,k)\leq kd-\lceil d/2\rceil$ for $k\geq 2$, and showed that $\delta(d,2)=\lfloor 3d/2\rfloor$. Pursuing Del Pia and Michini's approach, Deza and Pournin [5] showed that $\delta(d,k)\leq kd-\lceil 2d/3\rceil-(k-3)$ for $k\geq 3$, and that $\delta(4,3)=8$. The determination of $\delta(2,k)$ was investigated independently in the early nineties by Thiele [8], Balog and Bárány [2], and Acketa and Žunić [1]. Deza, Manoussakis, and Onn [4] showed that $\delta(d,k)\geq \lfloor (k+1)d/2\rfloor$ for all $k\leq 2d-1$ and proposed Conjecture 1.1.

Conjecture 1.1. *$\delta(d,k) \leq \lfloor (k+1)d/2 \rfloor$, and $\delta(d,k)$ is achieved, up to translation, by a Minkowski sum of lattice vectors.*

In Section 2, we propose a computational framework which drastically reduces the search space for lattice (d,k)-polytopes achieving a large diameter. Applying this framework to $(d,k) = (3,4)$ and $(3,5)$, we determine in Section 3 that $\delta(3,4) = 7$ and $\delta(3,5) = 9$.

Theorem 1.2. *Conjecture 1.1 holds for $(d,k) = (3,4)$ and $(3,5)$; that is, $\delta(3,4) = 7$ and $\delta(3,5) = 9$, and both diameters are achieved, up to translation, by a Minkowski sum of lattice vectors*

Note that Conjecture 1.1 holds for all known values of $\delta(d,k)$ given in Table 1, and hypothesizes, in particular, that $\delta(d,3) = 2d$. The new entries corresponding to $(d,k) = (3,4)$ and $(3,5)$ are entered in bold.

		k									
	$\delta(d,k)$	1	2	3	4	5	6	7	8	9	10
	1	1	1	1	1	1	1	1	1	1	1
	2	2	3	4	4	5	6	6	7	8	8
d	3	3	4	6	**7**	**9**					
	4	4	6	8							
	$\vdots$	$\vdots$	$\vdots$								
	d	d	$\lfloor \frac{3d}{2} \rfloor$								

Table 1: Largest diameter $\delta(d,k)$ over all lattice (d,k)-polytopes

2 Theoretical and Computational Framework

Since $\delta(2,k)$ and $\delta(d,2)$ are known, we consider in the remainder of the paper that $d \geq 3$ and $k \geq 3$. While the number of lattice (d,k)-lattice polytopes is finite, a brute force search is typically intractable, even for small instances. Theorem 2.1, which recalls conditions established in [5], allows to drastically reduce the search space.

Theorem 2.1. *For $d \geq 3$, let $d(u,v)$ denote the distance between two vertices u and v in the edge-graph of a lattice (d,k)-polytope P such that $d(u,v) = \delta(d,k)$. For $i = 1,\ldots,d$, let F_i^0, respectively F_i^k, denote the intersection of P with the facet of the cube $[0,k]^d$ corresponding to $x_i = 0$, respectively $x_i = k$. Then, $d(u,v) \leq \delta(d-1,k) + k$, and the following conditions are necessary for the inequality to hold with equality:*

(1) $u + v = (k,k,\ldots,k)$,

(2) *any edge of P with u or v as vertex is $\{-1,0,1\}$-valued,*

(3) *for* $i = 1, \ldots, d$, F_i^0, *respectively* F_i^k, *is a* $(d-1)$*-dimensional face of* P *with diameter* $\delta(F_i^0) = \delta(d-1,k)$, *respectively* $\delta(F_i^k) = \delta(d-1,k)$.

Thus, to show that $\delta(d,k) < \delta(d-1,k) + k$, it is enough to show that there is no lattice (d,k)-polytope admitting a pair of vertices (u,v) such that $d(u,v) = \delta(d,k)$ and the conditions (1), (2), and (3) are satisfied. The computational framework to determine, given (d,k), whether $\delta(d,k) = \delta(d-1,k)+k$ is outlined below and illustrated for $(d,k) = (3,4)$ or $(3,5)$.

Algorithm to determine whether $\delta(d,k) < \delta(d-1,k) + k$

Step 1*:* Initialization
Determine the set $\mathcal{F}$ *of all the lattice* $(d-1,k)$*-polytopes* P *such that* $\delta(P) = \delta(d-1,k)$*. For example, for* $(d,k) = (3,4)$*, the determination of all the* 335 *lattice* $(2,4)$*-polygons* P *such that* $\delta(P) = 4$ *is straightforward.*

Step 2*:* Symmetries
Consider, up to the symmetries of the cube $[0,k]^d$*, the possible entries for a pair of vertices* (u,v) *such that* $u+v = \{k,k,\ldots,k\}$*. For example, for* $(d,k) = (3,4)$*, the following 6 vertices cover all possibilities for* u *up to symmetry:* $(0,0,0), (0,0,1), (0,0,2), (0,1,1), (0,1,2)$, *and* $(0,2,2)$*, where* $v = (4,4,4) - u$*.*

Step 3*:* Shelling
For each of the possible pairs (u,v) *determined during Step* 2*, consider all possible ways for* $2d$ *elements of the set* $\mathcal{F}$ *determined during Step 1 to form the* $2d$ *facets of* P *lying on a facet of the cube* $[0,k]^d$*. For example, for* $(d,k) = (3,4)$ *and* $u = (0,0,0)$*, we must find* 6 *elements of* $\mathcal{F}$*,* 3 *with* $(0,0)$ *as a vertex, and* 3 *with* $(4,4)$ *as a vertex. In addition, if an edge of an element of* $\mathcal{F}$ *with* u *or* v *as vertex is not* $\{-1,0,1\}$*-valued, this element is disregarded.*

Note that since the choice of an element of $\mathcal{F}$ *defines the vertices of* P *belonging to a facet of the cube* $[0,k]^d$*, the choice for the next element of* $\mathcal{F}$ *to form a shelling is significantly restricted. In addition, if the set of vertices and edges belonging to the current elements of* $\mathcal{F}$ *considered for a shelling includes a path from* u *to* v *of length at most* $\delta(d-1,k)+k-1$*, a shortcut between* u *and* v *exists and the last added elements of* $\mathcal{F}$ *can be disregarded.*

Step 4. Inner points
For each choice of $2d$ *elements of* $\mathcal{F}$ *forming a shelling obtained during Step* 3*, consider the* $\{1,2,\ldots,k-1\}$*-valued points not in the convex hull of the vertices of the* $2d$ *elements of* $\mathcal{F}$ *forming a shelling. Each such* $\{1,2,\ldots,k-1\}$*-valued point is considered as a potential vertex of* P *in a binary tree. If the current set of edges includes a path from* u *to* v *of length at most* $\delta(d-1,k)+k-1$*, a shortcut between* u *and* v *exists and the corresponding node of the binary tree can be disregarded, and the the binary tree is pruned at this node.*

A convex hull and diameter computation are performed for each node of the obtained binary tree. If there is a node yielding a diameter of $\delta(d-1,k)+k$ we can conclude that $\delta(d,k)=\delta(d-1,k)+k$. Otherwise, we can conclude that $\delta(d,k)<\delta(d-1,k)+k$. For example, for $(d,k)=(3,5)$, no choice of 6 elements of $\mathcal{F}$ forming a shelling such that $d(u,v)\geq 10$ exist, and thus Step 4 is not executed.

3 Computational Results

For $(d,k)=(3,4)$, a shelling exists for which path lengths are not decidable by the algorithm without convex hull computations. However, this shelling only achieves a diameter of 7. For $(d,k)=(3,5)$ the algorithm stops at Step 3, as there is no combination of 6 elements of $\mathcal{F}$ which form a shelling such that $d(u,v)\geq\delta(2,5)+5$. Thus, no convex hull computations are required for $(d,k)=(3,5)$. A shortcut from u to v is typically found early on in the shelling, which leads to the algorithm terminating quickly. Run on a 2009 Intel® Core™2 Duo 2.20GHz CPU, the algorithm is able to terminate for $(d,k)=(3,4)$ and $(3,5)$ in under a minute. Consequently, $\delta(3,4)<8$ and $\delta(3,5)<10$. Since the Minkowski sum of $(1,0,0),(0,1,0),(0,0,1),(0,1,1),(1,0,1),(1,1,0)$, and $(1,1,1)$ forms a lattice $(3,4)$-polytope with diameter 7, we conclude that $\delta(3,4)=7$. Similarly, since the Minkowski sum of $(1,0,0),(0,1,0),(0,0,1)$, $(0,1,1),(1,0,1),(1,1,0),(0,1,-1),(1,0,-1)$, and $(1,-1,0)$ forms, up to translation, a lattice $(3,5)$-polytope with diameter 9, we conclude that $\delta(3,5)=9$. Computations for additional values of $\delta(d,k)$ are currently underway. In particular, the same algorithm may determine whether $\delta(d,k)=\delta(d-1,k)+k$ or $\delta(d-1,k)+k-1$ for $(d,k)=(5,3)$ and $(4,4)$ provided the set of all lattice $(d-1,k)$-polytopes achieving $\delta(d-1,k)$ is determined for $(d,k)=(5,3)$ and $(4,4)$. Similarly, the algorithm could be adapted to determine whether $\delta(d,k)<\delta(d-1,k)+k-1$ provided the set of all lattice $(d-1,k)$-polytopes achieving $\delta(d-1,k)$ or $\delta(d-1,k)-1$ is determined. For example, the adapted algorithm may determine whether $\delta(3,6)=10$.

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