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GENERALIZED SUPPORT SET INVARIANCY SENSITIVITY ANALYSIS

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Abstract. Support set invariancy sensitivity analysis deals with finding the range of the parameter variation where there are optimal solutions with the same positive variables for all parameter values throughout this range. This approach to sensitivity analysis has been studied for Linear Optimization (LO) and Convex Quadratic Optimization (CQO) problems, when they are in standard form. In practice, most problems are in *general form*, in addition to nonnegative variables and equalities, they include free variables and inequalities. Though, the LO problem in general form can be converted into the standard form, but this transforming changes the meaning of the support set invariancy sensitivity analysis.

In this paper, we consider the primal and dual LO problems in general form and investigate support set invariancy sensitivity analysis for them. We present computable LO problems to identify the associated support set invariancy intervals for LO problem and investigate their relationship to support set invariancy intervals for LO problem in standard form.

1. **Introduction.** The perturbed primal LO problem in general form can be defined as:

$$LP(\triangle b, \triangle c, \epsilon) \qquad \begin{array}{ll} \min & (c^{(1)} + \epsilon \triangle c^{(1)})^T x^{(1)} + (c^{(2)} + \epsilon \triangle c^{(2)})^T x^{(2)} \\ \text{s.t.} & A_{11} x^{(1)} + A_{12} x^{(2)} = b^{(1)} + \epsilon \triangle b^{(1)} \\ & A_{21} x^{(1)} + A_{22} x^{(2)} \geq b^{(2)} + \epsilon \triangle b^{(2)} \\ & x^{(1)} \text{ free}, \ x^{(2)} \geq 0, \end{array}$$

where, for $i, j = 1, 2$, matrices $A_{ij} \in \mathbb{R}^{m_i \times n_j}$, and vectors $c^{(j)}, \triangle c^{(j)} \in \mathbb{R}^{n_j}$ and $b^{(i)}, \triangle b^{(i)} \in \mathbb{R}^{m_i}$ are fixed data and $x^{(j)} \in \mathbb{R}^{n_j}$ are unknown vectors. Simply let $b = ({b^{(1)}}^T, {b^{(2)}}^T)^T$, $c = ({c^{(1)}}^T, {c^{(2)}}^T)^T$, $\triangle b = ({\triangle b^{(1)}}^T, {\triangle b^{(2)}}^T)^T$ and $\triangle c = ({\triangle c^{(1)}}^T, {\triangle c^{(2)}}^T)^T$. We refer to $\triangle b$ and $\triangle c$ as perturbation vectors. In special cases, one of the vectors $\triangle b$ and $\triangle c$ might be zero, or all but one of the components are zero. Observe that for $n_1 = m_2 = 0$, the problem $LP(\triangle b, \triangle c, \epsilon)$ is in standard form, for $n_1 = m_1 = 0$, it is in canonical form, and for only $n_1 = 0$, this problem is in

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mixed form. For parameter value $\epsilon = 0$, problem $LP(\triangle b, \triangle c, \epsilon)$ is an unperturbed primal LO problem and we denote it shortly by $LP = LP(\triangle b, \triangle c, 0)$. Throughout the paper, "*general LO problem*" is referred to the case when primal and dual LO problems are in general form containing free variables and inequalities in addition to nonnegative variables and equalities. Analogously, the term "*standard LO problem*" refers to the case when the primal and dual LO problems are in standard form.

Answering the question "What does happen to optimal solutions when perturbation occurs in the input data?" was one of the first goals for optimizers [13] soon after the simplex method was introduced [1]. The related subject area is known as *parametric programming* and *sensitivity analysis*.

A classification of sensitivity analysis for LO was introduced by Koltai and Terlaky [12]. They categorized the sensitivity analysis in three types. In Type I sensitivity analysis (*Basis Invariancy Sensitivity Analysis*) the aim is to find the range of parameter variation, where the given optimal basic solution remains optimal. This kind of sensitivity analysis is based on simplex methods and it is known as *classic* sensitivity analysis [11, 13]. A major weak point in basis invariancy sensitivity analysis is that, when the problem has multiple optimal and thus degenerate solutions (what frequently happens in practice), different methods lead to different optimal basic solutions and consequently, different and confusing optimality ranges are obtained [10, 14]. Type II sensitivity analysis (*Support Set Invariancy Sensitivity Analysis*) aims to identify the range of parameter variation, where for all parameter values in this range, the perturbed problem still has an optimal solution with the same set of positive variables that the given optimal solution of the unperturbed problem has. Finally, Type III sensitivity analysis (*Optimal Partition Invariancy Sensitivity Analysis*) is concerned with the behavior of the optimal value function under variation of the input data. All results in optimal partition invariancy sensitivity analysis are based on the assumption that the LO problem is in standard form [4, 8, 14].

Recently, Ghaffari and Terlaky [5, 6] investigated support set invariancy sensitivity analysis for primal LO and CQO problems by focusing on the support set of primal optimal solutions. They referred to the associated intervals as *invariant support set* intervals and denoted it by $\Upsilon_L(\triangle b, \triangle c)$ for LO. Moreover, Ghaffari et al. [3] studied the support set invariancy sensitivity analysis for dual LO and CQO problems by focusing on the support set of slack variables in dual optimal solutions. They referred to the associated intervals as *invariant active constraint set* interval and denoted it by $\Gamma_L(\triangle b, \triangle c)$ for LO. The case when, the primal optimal solution has invariant support set and the dual problem has invariant active constraint set, is also investigated by the authors. They referred to the associated intervals as *invariant set* interval and denoted it by $\Theta_L(\triangle b, \triangle c)$. All approaches in [3, 5, 6] are based on the assumption that the primal LO problem is in standard form. However, optimization problems are usually not in standard form in practice. They might include free variables and inequalities in addition to nonnegative variables and equalities.

Recall that the goal of support set invariancy sensitivity analysis for primal LO problems, is finding the range of parameter variation so that the primal perturbed problem has an optimal solution with the same set of positive variables that the given primal optimal solution of the unperturbed problem has [5]. An economic interpretation of support set invariancy sensitivity analysis for primal LO problems might be as: "The manager wants to maintain the active production lines active after any change in the markets and he is not willing to activate any inactive production lines." Meanwhile, converting problem LP to the standard form and applying the method that was introduced in [5] for LO problems in standard form, loses the desired objective of support set invariancy sensitivity analysis. Thus, this misusing of the methods leads to incorrect results for the original purpose and obtained results are not those we need. However, there is still an alternate interpretation for the results obtained that way. For instance, let the LO problem is in canonical form and we convert it to the standard form by adding slack variables. An interpretation of the results that are obtained by applying the method introduced in [5] on the converted LO problem, might be as: "For any ϵ in the obtained interval, the manager keeps the active production lines active, without launching new production lines. Moreover, he uses the whole amount of the sources that have been used before regardless the changes in the market and he has always some kind of resources in storage as he had before."

Meanwhile, the manager may be interested in finding the range of parameter variation with only one of the following goals:

- (**i**) Finding the range of parameter variation, where for any parameter value in this range, *only* the positive variables in the given optimal solution of the unperturbed problem are positive in an optimal solution of the perturbed problem, *without caring* what occurs with the constraints. In other words, the manager is not worrying about the source and inventory amounts but due to possible side expenses, his main concern is not to set up new production plans and not to close any active ones.
- (**ii**) Finding the range of parameter variation, where for any parameter value in this range, *only* the active constraints for a given optimal solution of the unperturbed problem remain active for an optimal solution of the perturbed problem, *without caring* what happens to optimal solutions. In other words, the manager has no problem with activating any new production plan or closing off any active production line, but his main concern is how to circumvent the possible supply and inventory problems that the market may imply.

In this paper, we are interested in studying sensitivity analysis with the aforementioned goals in (**i**) and/or (**ii**). The paper is organized as follows. Section 2 contains some necessary concepts. In Section 3, we define rigorously the support set sensitivity analysis for problem LP and continue by investigating some fundamental properties of the related intervals. The simultaneous perturbation case, when variation occurs in both the Right Hand Side (RHS) and the Objective Function Coefficient (OFC) data of LP, is considered and the behavior of the optimal value function on these intervals is studied. Auxiliary LO problems are presented in this section that allow us to identify the associated intervals. Special cases, when either $\triangle b$ or $\triangle c$ is a zero vector, are investigated too. A simple example is presented in Section 4 to illustrate our methods. The closing section contains the summary of the results, as well as the directions of future work.

2. **Preliminaries.**

Recall the problem $LP(\triangle b, \triangle c, \epsilon)$. Its dual is formulated as

$$LD(\triangle b, \triangle c, \epsilon) \qquad \begin{array}{ll} \max & (b^{(1)} + \epsilon\triangle b^{(1)})^T y^{(1)} + (b^{(2)} + \epsilon\triangle b^{(2)})^T y^{(2)} \\ \text{s.t.} & A_{11}^T y^{(1)} + A_{21}^T y^{(2)} = c^{(1)} + \epsilon\triangle c^{(1)} \\ & A_{12}^T y^{(1)} + A_{22}^T y^{(2)} \leq c^{(2)} + \epsilon\triangle c^{(2)} \\ & y^{(1)} \text{ free},\ y^{(2)} \geq 0, \end{array}$$

where $y_i \in \mathbb{R}^{m_i}$, for $i = 1, 2$, are unknown vectors. For the parameter value $\epsilon = 0$, we denote this problem shortly by $LD = LD(\triangle b, \triangle c, 0)$.

In the sequel, some fundamental concepts from standard LO problem are generalized to the general LO problem. Any vector $x = ({x^{(1)}}^T, {x^{(2)}}^T)^T$ with $x^{(2)} \geq 0$ satisfying the constraints of LP is called a *primal feasible* solution and any vector $y = ({y^{(1)}}^T, {y^{(2)}}^T)^T$ with $y^{(2)} \geq 0$ satisfying the constraints of LD is called a *dual feasible* solution. We refer to the index sets $\{1, \ldots, n_2\}$ and $\{1, \ldots, m_2\}$ as *variables index set* and *constraints index set* of the problem LP, respectively.

Observe that the inequalities in problems LP and LD can be replaced by $A_{21}x^{(1)} + A_{22}x^{(2)} - r = b^{(2)}$ and $A_{12}^T y^{(1)} + A_{22}^T y^{(2)} + s = c^{(2)}$, respectively, where $r \in \mathbb{R}^{m_2}$ and $s \in \mathbb{R}^{n_2}$ are unknown nonnegative vectors. We refer to r and s as *reduced cost* and *slack* vectors, respectively. In this way, primal and dual feasible solutions can be denoted by (x, r) and (y, s), respectively. For any primal-dual feasible solution (x, r, y, s), the weak duality property $b^T y \leq c^T x$ holds, where $b^T = ({b^{(1)}}^T, {b^{(2)}}^T)^T$ and $c^T = ({c^{(1)}}^T, {c^{(2)}}^T)^T$. If $b^T y = c^T x$ (*strong duality*), then the feasible solutions (x, r) and (y, s) are primal and dual optimal solutions of problems LP and LD, respectively. Consequently, for a primal-dual optimal solution $(x^{(1)*}, x^{(2)*}, r^*, y^{(1)*}, y^{(2)*}, s^*)$, we have ${s^*}^T x^{(2)*} + {r^*}^T y^{(2)*} = 0$. Considering the nonnegativity of variables $x^{(2)*}$, $y^{(2)*}$, r^* and s^*, the optimality property can be rewritten as $s_j^* x_j^{(2)*} = 0$ for $j \in \{1, \ldots, n_2\}$ and $r_i^* y_i^{(2)*} = 0$ for $i \in \{1, \ldots, m_2\}$. Clearly speaking, for a primal-dual optimal solution $(x^{(1)*}, x^{(2)*}, r^*, y^{(1)*}, y^{(2)*}, s^*)$, the vectors $x^{(2)*}$ and s^*, and the vectors $y^{(2)*}$ and r^* are complementary.

The *support* set of a nonnegative vector v is defined as $\sigma(v) = \{i | v_i > 0\}$. Considering this notation, the optimality condition implies the following equalities:

$$\sigma(x^{(2)*}) \cap \sigma(s^*) = \emptyset \text{ and } \sigma(y^{(2)*}) \cap \sigma(r^*) = \emptyset, \tag{1}$$

where $(x^{(1)*}, x^{(2)*}, r^*, y^{(1)*}, y^{(2)*}, s^*)$ is a primal-dual optimal solution of problems LP and LD. A complementary (optimal) solution $(x^{(1)*}, x^{(2)*}, r^*, y^{(1)*}, y^{(2)*}, s^*)$ is *primal strictly complementary*, if ${s^*}^T x^{(2)*} = 0$ with $s^* + x^{(2)*} > 0$ and it is *dual strictly complementary*, if ${r^*}^T y^{(2)*} = 0$ with $r^* + y^{(2)*} > 0$. If a primal-dual optimal solution $(x^{(1)*}, x^{(2)*}, r^*, y^{(1)*}, y^{(2)*}, s^*)$ is primal and dual strictly complementary optimal solution, we say shortly that it is a strictly complementary solution. For a strictly complementary optimal solution $(x^{(1)*}, x^{(2)*}, r^*, y^{(1)*}, y^{(2)*}, s^*)$, besides the complementarity condition (1), the following relations hold:

$$\sigma(x^{(2)*}) \cup \sigma(s^*) = \{1, \ldots, n_2\} \text{ and } \sigma(y^{(2)*}) \cup \sigma(r^*) = \{1, \ldots, m_2\}. \tag{2}$$

It should be mentioned that for primal (dual) strictly complementary solution, only the first (second) equality in (2) holds. By the Goldman-Tucker Theorem [7], the existence of a strictly complementary optimal solution of problems LP and LD is guaranteed if these problems are feasible.

Let $\mathcal{LP}(\triangle b, \triangle c, \epsilon)$ and $\mathcal{LD}(\triangle b, \triangle c, \epsilon)$ be feasible sets of problems $LP(\triangle b, \triangle c, \epsilon)$ and $LD(\triangle b, \triangle c, \epsilon)$, respectively. Further, let $\mathcal{LP}^*(\triangle b, \triangle c, \epsilon)$ and $\mathcal{LD}^*(\triangle b, \triangle c, \epsilon)$ denote their optimal solution sets, correspondingly. When $\epsilon = 0$, we denote the feasible sets shortly by $\mathcal{LP} = \mathcal{LP}(\triangle b, \triangle c, 0)$ and $\mathcal{LD} = \mathcal{LD}(\triangle b, \triangle c, 0)$. Analogously, $\mathcal{LP}^* = \mathcal{LP}^*(\triangle b, \triangle c, 0)$ and $\mathcal{LD}^* = \mathcal{LD}^*(\triangle b, \triangle c, 0)$, i.e.,

$$\begin{aligned} \mathcal{LP}^* &= \{(x^*, r^*) |\ (x^*, r^*) \text{ is an optimal solution in } \mathcal{LP}\}, \\ \mathcal{LD}^* &= \{(y^*, s^*) |\ (y^*, s^*) \text{ is an optimal solution in } \mathcal{LD}\}. \end{aligned}$$

Considering (1) and (2), one can define the partition

$$\begin{aligned}\mathcal{B}_V^x &= \{j : x_j^{(2)*} > 0,\ \forall j \in \{1,\dots,n_2\} \text{ for some } (x^{(1)*}, x^{(2)*}, r^*) \in \mathcal{LP}^*\},\\ \mathcal{N}_V^s &= \{j : s_j^* > 0,\ \forall j \in \{1,\dots,n_2\} \text{ for some } (y^{(1)*}, y^{(2)*}, s^*) \in \mathcal{LD}^*\},\end{aligned}$$

for the variables index set $\{1,\dots,n_2\}$ and the partition

$$\begin{aligned}\mathcal{B}_C^y &= \{i : y_i^{(2)*} > 0,\ \forall i \in \{1,\dots,m_2\} \text{ for some } (y^{(1)*}, y^{(2)*}, s^*) \in \mathcal{LD}^*\},\\ \mathcal{N}_C^r &= \{i : r_i^* > 0,\ \forall i \in \{1,\dots,m_2\} \text{ for some } (x^{(1)*}, x^{(2)*}, r^*) \in \mathcal{LP}^*\}.\end{aligned}$$

for the constraints index set $\{1,\dots,m_2\}$. Roughly speaking, in any primal optimal solution $(x^{(1)*}, x^{(2)*}, r^*)$, any component of $x^{(2)*}$ with its index in $\mathcal{N}_V^s$, is always zero. Moreover, for any inequality constraint with its index in $\mathcal{N}_C^r$, the corresponding component of the dual vector $y^{(2)*}$ is always zero in any optimal solution. We denote these partitions by $\pi_V = (\mathcal{B}_V^x, \mathcal{N}_V^s)$ and $\pi_C = (\mathcal{B}_C^y, \mathcal{N}_C^r)$ and refer to them as *variables optimal partition* and *constraints optimal partition*, respectively. The uniqueness of these partitions is a direct consequence of the convexity of the optimal solution sets $\mathcal{LP}^*$ and $\mathcal{LD}^*$.

Recall that, when in problem LP, the relation $n_1 = m_2 = 0$ holds, then this problem reduces to the standard form. In this case, $r = 0$ holds in all primal optimal solutions in $\mathcal{LP}^*$ and $y^{(2)}$ does not exist in all optimal solutions of the dual problem LD. Thus, the constraints index set is empty in this case and so the sets $\mathcal{B}_C^y$ and $\mathcal{N}_C^r$ are. Consequently, $\pi_V = (\mathcal{B}_V^x, \mathcal{N}_V^s)$ reduces to the usual optimal partition that is defined in optimization contexts for the standard LO problem (see e.g., [5]). Moreover, the meaning of the strictly complementary optimal solution for general LO problem is specialized to its usual concept in standard LO problem, and in this case, the strong duality property reduces to its common description in standard LO problem too (see e.g., [5]). In this paper, we show that the outlined results in support set invariancy sensitivity analysis for general LO problem, reduce to those results that were obtained for the standard LO problem in [3, 5].

Interior Point Methods (IPMs) solve optimization problems in polynomial time [14]. They start from a feasible (or an infeasible) interior point of the positive orthant and generate an interior solution near to optimal set. From that absolute positive optimal solution, by using a simple rounding procedure [9], a strictly complementary optimal solution of the general LO problem can be obtained in strongly polynomial time.

The optimal value function of the perturbed general LO problem is defined as:

$$\begin{aligned}\phi(\triangle b, \triangle c, \epsilon) &= (c^{(1)} + \epsilon\triangle c^{(1)})^T x^{(1)*} + (c^{(2)} + \epsilon\triangle c^{(2)})^T x^{(2)*}\\ &= (b^{(1)} + \epsilon\triangle b^{(1)})^T y^{(1)*} + (b^{(2)} + \epsilon\triangle b^{(2)})^T y^{(2)*},\end{aligned}$$

where $(x^{(1)*}(\epsilon), x^{(2)*}(\epsilon), r^*(\epsilon), y^{(1)*}(\epsilon), y^{(2)*}(\epsilon), s^*(\epsilon))$ is a primal-dual optimal solution of problems $LP(\triangle b, \triangle c, \epsilon)$ and $LD(\triangle b, \triangle c, \epsilon)$.

Observe that when $\triangle b$ and $\triangle c$ are nonzero vectors and the given primal problem is in standard form, i.e., when $n_1 = m_2 = 0$, the optimal value function $\phi(\triangle b, \triangle c, \epsilon)$ is a quadratic function, on the interval where the (variables) optimal partition is invariant [4]. This property of the optimal value function was also proved for the case when the given primal and dual optimal problems are in canonical form [8], i.e., when $n_1 = m_1 = 0$. It is also proved that on the invariancy interval the optimal value function is linear when either $\triangle b$ or $\triangle c$ is a zero vector [14, 8]. Here, we present a general statement referring to these properties of the optimal value function for the general LO problem.

Theorem 1. *Consider the problems LP and LD and let $\phi(\triangle b, \triangle c, \epsilon)$ denote the corresponding optimal value function. Moreover, let ϵ_1 and ϵ_2 be two real numbers for which the optimal partitions π_V and π_C are the same. Then, for all $\epsilon \in [\epsilon_1, \epsilon_2]$, these optimal partitions are invariant. Further,*
(i) The optimal value function is quadratic on $[\epsilon_1, \epsilon_2]$, when $\triangle b$ and $\triangle c$ do not vanish simultaneously.
(ii) The optimal value function is linear on $[\epsilon_1, \epsilon_2]$, when either $\triangle b$ or $\triangle c$ is a zero vector.

Proof. The proof of (i) is similar to the proof of Theorem 3.2 in [3] and the proofs of (ii) are analogous to the proofs of Theorems IV.50 and IV.51 in [14]. □

Remark 1. It is proved that the optimal value function is a continuous piecewise quadratic function on its domain for standard LO problems [4] and it is a continuous piecewise linear function when either $\triangle b$ or $\triangle c$ is a zero vector [2, 14]. One can generalize these facts to establish analogous statements for the general LO problems. We omit the details here.

Let Φ_{LP} denote the solution set of the following equations system:

$$\begin{array}{lllll} A_{11}x^{(1)} & +A_{12}x^{(2)} & & = & b^{(1)} + \epsilon\triangle b^{(1)} \\ A_{21}x^{(1)} & +A_{22}x^{(2)} & -r & = & b^{(2)} + \epsilon\triangle b^{(2)}, \end{array}$$

where $x^{(2)} \geq 0$ and $r \geq 0$. Analogously, let Φ_{LD} denote the solution set of the following equations system:

$$\begin{array}{lllll} A_{11}^T y^{(1)} & +A_{21}^T y^{(2)} & & = & c^{(1)} + \epsilon\triangle c^{(1)} \\ A_{12}^T y^{(1)} & +A_{22}^T y^{(2)} & +s & = & c^{(2)} + \epsilon\triangle c^{(2)}, \end{array}$$

where $y^{(2)} \geq 0$ and $s \geq 0$. Further, combining Φ_{LP} and Φ_{LD}, let Φ denote the solution set of the following equations system:

$$\begin{array}{llllllll} A_{11}x^{(1)} & +A_{12}x^{(2)} & & & & & = & b^{(1)} + \epsilon\triangle b^{(1)} \\ A_{21}x^{(1)} & +A_{22}x^{(2)} & & & -r & & = & b^{(2)} + \epsilon\triangle b^{(2)} \\ & & A_{11}^T y^{(1)} & +A_{21}^T y^{(2)} & & & = & c^{(1)} + \epsilon\triangle c^{(1)} \\ & & A_{12}^T y^{(1)} & +A_{22}^T y^{(2)} & & +s & = & c^{(2)} + \epsilon\triangle c^{(2)}, \end{array}$$

where $x^{(2)} \geq 0$, $r \geq 0$, $y^{(2)} \geq 0$ and $s \geq 0$. It should be mentioned that when the problem LP is in canonical form , i.e., when $n_1 = m_1 = 0$, we have $A = A_{22}$, $b = b^{(2)}$, $c = c^{(2)}$, $\triangle b = \triangle b^{(2)}$, $\triangle c = \triangle c^{(2)}$, $x = x^{(2)}$ and $y = y^{(2)}$. In this case, the set Φ reduces to the solution set of the follwong equations system:

$$\begin{array}{llllll} Ax & & -r & & = & b + \epsilon\triangle b \\ & A^T y & & +s & = & c + \epsilon\triangle c, \end{array}$$

where $x \geq 0$, $r \geq 0$, $y \geq 0$ and $s \geq 0$. We denote the solution set of these equations system by Φ_C.

Observe that in the feasible solution set $\mathcal{LP}^*(\triangle b, \triangle c, \epsilon)$, the parameter ϵ is considered to be a fixed parameter value and this set contains all vectors $(x(\epsilon), r(\epsilon))$ that satisfy the constraints of problem $LP(\triangle b, \triangle c, \epsilon)$. Meanwhile, in solution set Φ, ϵ is considered as an unknown and its smallest and biggest values (if exist) denote the domain of the optimal value function $\phi(\triangle b, \triangle c, \epsilon)$. Analogous discussion is valid for solution sets Φ_{LP} and Φ_{LD}.

The following theorem presents computable LO problems that allow us to identify the end points of an interval (might be a trivial singleton in special situations), where optimal partitions are invariant. It also establishes a relationship between these intervals in simultaneous and nonsimultaneous perturbation cases. The proofs are straightforward and they are omitted.

Theorem 2. *Consider the primal and dual LO problems LP and LD, respectively. Further, let $\pi_V = (\mathcal{B}_V^x, \mathcal{N}_V^s)$ and $\pi_C = (\mathcal{B}_C^y, \mathcal{N}_C^r)$ be the variables and constraints optimal partitions of problem LP, respectively. Then*

- *For $\triangle c = 0$, the optimal partitions π_V and π_C are invariant for any $\epsilon \in (\lambda_\ell^{LP}, \lambda_u^{LP})$, where λ_ℓ^{LP} and λ_u^{LP} are obtained by solving the following auxiliary LO problems, respectively:*

$$\begin{aligned}\lambda_\ell^{LP} &= \min\left\{\epsilon|(x^{(1)}, x^{(2)}, r, \epsilon) \in \Phi_{LP},\ \sigma(x^{(2)}) \subseteq \mathcal{B}_V^x, \sigma(r) \subseteq \mathcal{N}_C^r\right\},\\ \lambda_u^{LP} &= \max\left\{\epsilon|(x^{(1)}, x^{(2)}, r, \epsilon) \in \Phi_{LP},\ \sigma(x^{(2)}) \subseteq \mathcal{B}_V^x, \sigma(r) \subseteq \mathcal{N}_C^r\right\}.\end{aligned}$$

- *For $\triangle b = 0$, the optimal partitions π_V and π_C are invariant for any $\epsilon \in (\lambda_\ell^{LD}, \lambda_u^{LD})$, where λ_ℓ^{LD} and λ_u^{LD} are obtained by solving the following minimization and maximization problems, respectively:*

$$\begin{aligned}\lambda_\ell^{LD} &= \min\left\{\epsilon|(y^{(1)}, y^{(2)}, s, \epsilon) \in \Phi_{LD},\ \sigma(y^{(2)}) \subseteq \mathcal{B}_C^y, \sigma(s) \subseteq \mathcal{N}_V^s\right\},\\ \lambda_u^{LD} &= \max\left\{\epsilon|(y^{(1)}, y^{(2)}, s, \epsilon) \in \Phi_{LD},\ \sigma(y^{(2)}) \subseteq \mathcal{B}_C^y, \sigma(s) \subseteq \mathcal{N}_V^s\right\}.\end{aligned}$$

- *For nonzero perturbation vectors $\triangle b$ and $\triangle c$, the optimal partitions π_V and π_C are invariant for any $\epsilon \in (\lambda_\ell, \lambda_u)$, where*

$$(\lambda_\ell, \lambda_u) = (\lambda_\ell^{LP}, \lambda_u^{LP}) \cap (\lambda_\ell^{LD}, \lambda_u^{LD}).$$

Let us refer to the points outlined in Theorem 2 as *transition point*s and the interval between two consequent transition points as *Optimal Partitions Invariancy (OPI) interval*. According to Theorem 2, at least one of the optimal partitions π_V and π_C at a transition point differs from its counterpart for any parameter value on its immediate neighboring interval. Thus, any transition point is a proper OPI interval (a singleton). One may construct an algorithmic procedure to identify all transition points, as well as optimal partitions on all possible OPI intervals as it was done in [4] for standard LO problem.

The *actual* OPI interval is that OPI interval which contains the actual parameter value $\epsilon = 0$. It should be mentioned that the actual OPI interval might be the singleton $\{0\}$ when $\lambda_\ell = \lambda_u = 0$ (or $\lambda_\ell^{LP} = \lambda_u^{LP} = 0$, or $\lambda_\ell^{LD} = \lambda_u^{LD} = 0$).

Remark 2. It must be mentioned that when $n_1 = m_2 = 0$, the feasible solution sets of problems that were introduced in Theorem 2, reduce to the feasible solution sets of their counterpart problems that were given in [4] for the standard LO problem. Further, when $n_1 = m_1 = 0$ holds, then Theorem 2 provides auxiliary LO problems to identify the actual OPI interval, when problem LP is in canonical form.

Remark 3. It is worth mentioning that the concept of the OPI can be restricted for either variables OPI or constraints OPI. In this paper, we prefer to consider the general concept.

3. **Support set sensitivity analysis for general LO problem.** Let us redefine the support set invariancy sensitivity analysis for general LO problem exactly. The following definitions are based on primal problem LP. Because of symmetry, one can easily construct these definitions for the dual problem LD.

Definition 1. (**Support set invariancy sensitivity analysis for problem LP**) Consider the problem LP and let a primal optimal solution $(x^{(1)*}, x^{(2)*}, r^*)$ with $\sigma(x^{(2)*}) = P$ be given. In support set invariancy sensitivity analysis for problem LP, we want to find the range of parameter variation, where for any ϵ in this range, the perturbed problem $LP(\triangle b, \triangle c, \epsilon)$ has an optimal solution $(x^{(1)*}(\epsilon), x^{(2)*}(\epsilon), r^*(\epsilon))$ with $\sigma(x^{(2)*}(\epsilon)) = P$.

Analogous definition can be considered by focusing on the support set of the reduced price vector r.

Definition 2. (**Active constraint set invariancy sensitivity analysis for problem LP**) Consider the problem LP and let a primal optimal solution $(x^{(1)*}, x^{(2)*}, r^*)$ with $\sigma(r^*) = \widehat{P}$ be given. In active constraint set invariancy sensitivity analysis for problem LP, we want to find the range of parameter variation, where for any ϵ in this range, the perturbed problem $LP(\triangle b, \triangle c, \epsilon)$ has an optimal solution $(x^{(1)*}(\epsilon), x^{(2)*}(\epsilon), r^*(\epsilon))$ with $\sigma(r^*(\epsilon)) = \widehat{P}$.

One may interested in combination of the goals in Definitions 1 and 2. The following statement defines this combined goal.

Definition 3. (**Characteristics invariancy**[1] **sensitivity analysis for problem LP**) Consider the problem LP and let a primal optimal solution $(x^{(1)*}, x^{(2)*}, r^*)$ with $\sigma(x^{(2)*}) = P$ and $\sigma(r^*) = \widehat{P}$ be given. In characteristics invariancy sensitivity analysis for problem LP, we want to find the range of parameter variation, where for any ϵ in this range, the perturbed problem $LP(\triangle b, \triangle c, \epsilon)$ has an optimal solution $(x^{(1)*}(\epsilon), x^{(2)*}(\epsilon), r^*(\epsilon))$ with $\sigma(x^{(2)*}(\epsilon)) = P$ and $\sigma(r^*(\epsilon)) = \widehat{P}$.

Having a primal optimal solution $(x^{(1)*}, x^{(2)*}, r^*)$, with $\sigma(x^{(2)*}) = P$ and $\sigma(s^*) = \widehat{P}$, a partition (P, Z) of the variables index set $\{1, \ldots, n_2\}$ and a partition $(\widehat{P}, \widehat{Z})$ of the constraints index set $\{1, \ldots, m_2\}$ can be defined, where $Z = \{1, \ldots, n_2\} \setminus P$ and $\widehat{Z} = \{1, \ldots, m_2\} \setminus \widehat{P}$. We refer to the partition (P, Z) as *Invariant Support Set* (ISS) partition and to the partition $(\widehat{P}, \widehat{Z})$ as *Invariant Active Constraint Set* (IACS) partition. If a problem LP has an optimal solution $(x^{(1)*}, x^{(2)*}, r^*)$ with $\sigma(x^{(2)*}) = P$, we say that this problem satisfies the *Generalized Invariant Support Set* (GISS) property w.r.t. the ISS partition (P, Z). On the other hand, if in the optimal solution $(x^{(1)*}, x^{(2)*}, r^*)$, the relation $\sigma(r^*) = \widehat{P}$ holds, it is said that the problem LP satisfies the *Generalized Invariant Active Constraint set* (GIACS) property w.r.t. the IACS partition $(\widehat{P}, \widehat{Z})$. Moreover, if this problem satisfies both the GISS and GIACS properties, we say that it has the *Generalized Invariant Characteristic* (GIC) property w.r.t. the partitions (P, Z) and $(\widehat{P}, \widehat{Z})$. The word "*generalized*" is referred to the observation that the given primal LO problem is in general form. To maintain the consistency of notations with [3, 5, 6], let $\Upsilon^G_L(\triangle b, \triangle c)$, $\Gamma^G_L(\triangle b, \triangle c)$ and $\Theta^G_L(\triangle b, \triangle c)$ denote the range of parameter variation associated with Definitions 1, 2 and 3, respectively. We refer to them as GISS,

[1] This terminology was used first by H.J. Greenberg.

GIACS and GIC sets, correspondingly. Analogous notations are used when either $\triangle b$ or $\triangle c$ is a zero vector.

3.1. **Fundamental properties.** Here, some fundamental properties of the GISS, GIACS and GIC sets are studied. It is obvious that these sets are not empty, because $LP(\triangle b, \triangle c, 0) = LP$ that satisfies the GISS and GIACS properties, and consequently it satisfies the GIC property. Let us state the following lemma that refers to the convexity property of these sets.

Lemma 1. *(i) Let the problem LP satisfy the GISS property w.r.t. the ISS partition (P, Z). Then, $\Upsilon_L^G(\triangle b, \triangle c)$ is a convex set.*
(ii) Let the problem LP satisfy the GIACS property w.r.t. the IACS partition $(\widehat{P}, \widehat{Z})$. Then, $\Gamma_L^G(\triangle b, \triangle c)$ is a convex set.
(iii) Let the problem LP has the GIC property w.r.t. the ISS partition (P, Z) and the IACS partition $(\widehat{P}, \widehat{Z})$. Then, $\Theta_L^G(\triangle b, \triangle c)$ is a convex set.

Proof. We only provide a proof for (i) and the proofs for (ii) and (iii) go analogously. Let a primal-dual optimal solution $(x^{(1)*}, x^{(2)*}, r^*, y^{(1)*}, y^{(2)*}, s^*)$ of problems LP and LD be given with $\sigma(x^{(2)*}) = P$. For $\epsilon_1, \epsilon_2 \in \Upsilon_L^G(\triangle b, \triangle c)$, let $(x^{(1)*}(\epsilon_1), x^{(2)*}(\epsilon_1), r^*(\epsilon_1), y^{(1)*}(\epsilon_1), y^{(2)*}(\epsilon_1), s^*(\epsilon_1))$ and $(x^{(1)*}(\epsilon_2), x^{(2)*}(\epsilon_2), r^*(\epsilon_2),$ $y^{(1)*}(\epsilon_2), y^{(2)*}(\epsilon_2), s^*(\epsilon_2))$ be given primal-dual optimal solutions of problems $LP(\triangle b, \triangle c, \epsilon)$ and $LD(\triangle b, \triangle c, \epsilon)$ at ϵ_1 and ϵ_2, respectively. By the assumption, we have $\sigma(x^{(2)*}(\epsilon_1)) = \sigma(x^{(2)*}(\epsilon_2)) = P$. For any $\epsilon \in (\epsilon_1, \epsilon_2)$ with $\theta = \frac{\epsilon_2 - \epsilon}{\epsilon_2 - \epsilon_1} \in (0, 1)$, we define

$$
\begin{aligned}
x^{(1)*}(\epsilon) &= \theta x^{(1)*}(\epsilon_1) + (1-\theta) x^{(1)*}(\epsilon_2), \\
x^{(2)*}(\epsilon) &= \theta x^{(2)*}(\epsilon_1) + (1-\theta) x^{(2)*}(\epsilon_2), \qquad (3)\\
r^*(\epsilon) &= \theta r^*(\epsilon_1) + (1-\theta) r^*(\epsilon_2), \qquad (4)\\
y^{(1)*}(\epsilon) &= \theta y^{(1)*}(\epsilon_1) + (1-\theta) y^{(1)*}(\epsilon_2), \\
y^{(2)*}(\epsilon) &= \theta y^{(2)*}(\epsilon_1) + (1-\theta) y^{(2)*}(\epsilon_2), \qquad (5)\\
s^*(\epsilon) &= \theta s^*(\epsilon_1) + (1-\theta) s^*(\epsilon_2). \qquad (6)
\end{aligned}
$$

It is easy to verify that $(x^{(1)*}(\epsilon), x^{(2)*}(\epsilon), r^*(\epsilon)) \in \mathcal{LP}(\triangle b, \triangle c, \epsilon)$ and $(y^{(1)*}(\epsilon),$ $y^{(2)*}(\epsilon), s^*(\epsilon)) \in \mathcal{LD}(\triangle b, \triangle c, \epsilon)$. Moreover, the optimality property $s^{*T}(\epsilon) x^{(2)*}(\epsilon) = r^{*T}(\epsilon) y^{(2)*}(\epsilon) = 0$ immediately follows from definitions (3)-(6). We also have $\sigma(x^{(2)*}(\epsilon)) = P$, i.e., $\epsilon \in \Upsilon_L^G(\triangle b, \triangle c)$ that completes the proof. □

Lemma 1 proves that the sets $\Upsilon_L^G(\triangle b, \triangle c)$, $\Gamma_L^G(\triangle b, \triangle c)$ and $\theta_L^G(\triangle b, \triangle c)$ are intervals of the real line that contain the actual parameter value $\epsilon = 0$. Therefore, we refer to them as "interval" instead of "set" throughout the paper. As we will see later, these intervals might be open, closed or half-closed intervals.

Let us investigate the behavior of the optimal value function on the GISS, GIACS and GIC intervals. The following theorem proves that the optimal value function $\phi(\triangle b, \triangle c, \epsilon)$ is quadratic on the closure of these intervals when both $\triangle b$ and $\triangle c$ are nonzero vectors. The proof is a direct generalization of the proof of Theorem 3.2 in [3] and it is omitted.

Theorem 3. *The optimal value function $\phi(\triangle b, \triangle c, \epsilon)$ is quadratic on the closure of the intervals $\Upsilon_L^G(\triangle b, \triangle c)$, $\Gamma_L^G(\triangle b, \triangle c)$ and $\Theta_L^G(\triangle b, \triangle c)$, when both $\triangle b$ and $\triangle c$ are nonzero perturbation vectors.*

Corollary 1. *The optimal value function $\phi(\triangle b, \triangle c, \epsilon)$ is linear on the closure of the intervals $\Upsilon_L^G(\triangle b, \triangle c)$, $\Gamma_L^G(\triangle b, \triangle c)$ and $\Theta_L^G(\triangle b, \triangle c)$, when either $\triangle b$ or $\triangle c$ is a zero vector.*

Considering Theorems 1 and 3 and Corollary 1, one can easily derive the following relationship between the intervals $\Upsilon_L^G(\triangle b, \triangle c)$, $\Gamma_L^G(\triangle b, \triangle c)$, $\Theta_L^G(\triangle b, \triangle c)$ and the actual OPI interval.

Corollary 2. *Assume that perturbation occurs in the RHS and/or OFC data of problem LP. Then, the intervals $\Upsilon_L^G(\triangle b, \triangle c)$, $\Gamma_L^G(\triangle b, \triangle c)$ and $\Theta_L^G(\triangle b, \triangle c)$ can not cover more than the closure of the actual OPI interval.*

3.2. **Identifying the GISS, GIACS and GIC intervals.** In this subsection we present auxiliary LO problems that enable us to identify the GISS, GIACS and GIC intervals. Special cases when either $\triangle b$ or $\triangle c$ is a zero vector are studied, as well as the cases when the general LO problem reduces to standard and canonical forms.

3.2.1. *Identifying the GISS intervals.* The following theorem presents two auxiliary LO problems that enable us to identify ϵ_ℓ and ϵ_u, i.e., the end points of $\overline{\Upsilon}_L^G(\triangle b, \triangle c)$, where $\overline{\Upsilon}_L^G(\triangle b, \triangle c)$ denotes the closure of the GISS interval $\Upsilon_L^G(\triangle b, \triangle c)$.

Theorem 4. *Let $(x^{(1)*}, x^{(2)*}, r^*)$ be a primal optimal solution of problems LP, where $\sigma(x^{(2)*}) = P$. Further, let $\pi_C = (\mathcal{B}_C^y, \mathcal{N}_C^r)$ be the constraints optimal partition of problem LP. Then, ϵ_ℓ and ϵ_u, the end points of $\overline{\Upsilon}_L^G(\triangle b, \triangle c)$, can be obtained by solving the following auxiliary LO problems, respectively:*

$$\begin{aligned}\epsilon_\ell &= \min\{\epsilon \mid (x^{(1)}, x^{(2)}, r, y^{(1)}, y^{(2)}, s, \epsilon) \in \Phi,\\ &\qquad \sigma(y^{(2)}) \subseteq \mathcal{B}_C^y,\ \sigma(r) \subseteq \mathcal{N}_C^r,\ \sigma(x^{(2)}) \subseteq P,\ \sigma(s) \subseteq Z\}, \quad (7)\\ \epsilon_u &= \max\{\epsilon \mid (x^{(1)}, x^{(2)}, r, y^{(1)}, y^{(2)}, s, \epsilon) \in \Phi,\\ &\qquad \sigma(y^{(2)}) \subseteq \mathcal{B}_C^y,\ \sigma(r) \subseteq \mathcal{N}_C^r,\ \sigma(x^{(2)}) \subseteq P,\ \sigma(s) \subseteq Z\}. \quad (8)\end{aligned}$$

Proof. Recall that $\epsilon_\ell \leq 0 \leq \epsilon_u$ and if $\epsilon_\ell = \epsilon_u = 0$, then the inclusion $[\epsilon_\ell, \epsilon_u] \subseteq \mathrm{int}(\Upsilon_L^G(\triangle b, \triangle c))$ follows immediately. Let us consider the case when at least one of ϵ_ℓ and ϵ_u is not zero. Without loss of generality, let $-\infty < \epsilon_\ell < 0$ and $\epsilon \in (\epsilon_\ell, 0)$ be given. Let $(\epsilon_\ell, x^{(1)*}(\epsilon_\ell), x^{(2)*}(\epsilon_\ell), r^*(\epsilon_\ell), y^{(1)*}(\epsilon_\ell), y^{(2)*}(\epsilon_\ell), s^*(\epsilon_\ell))$ be an optimal solution of the minimization problem (7). It is obvious that $\sigma(x^{(2)*}(\epsilon_\ell)) \subseteq P$.

Let us define

$$\begin{aligned}x^{(1)*}(\epsilon) &= \theta x^{(1)*}(\epsilon_\ell) + (1-\theta)x^{(1)*}, \quad (9)\\ x^{(2)*}(\epsilon) &= \theta x^{(2)*}(\epsilon_\ell) + (1-\theta)x^{(2)*}, \quad (10)\\ r^*(\epsilon) &= \theta r^*(\epsilon_\ell) + (1-\theta)r^*, \quad (11)\\ y^{(1)*}(\epsilon) &= \theta y^{(1)*}(\epsilon_\ell) + (1-\theta)y^{(1)*}, \quad (12)\\ y^{(2)*}(\epsilon) &= \theta y^{(2)*}(\epsilon_\ell) + (1-\theta)y^{(2)*}, \quad (13)\\ s^*(\epsilon) &= \theta s^*(\epsilon_\ell) + (1-\theta)s^*, \quad (14)\end{aligned}$$

where $\theta = \frac{\epsilon}{\epsilon_\ell} \in (0,1)$ and $(y^{(1)*}, y^{(2)*}, s^*)$ is a dual optimal solution of the problem LD. By construction,

$$\sigma(x^{(2)*}(\epsilon)) = P \text{ and } \sigma(s^*(\epsilon)) \subseteq Z. \quad (15)$$

One can easily verify that $(x^{(1)*}(\epsilon), x^{(2)*}(\epsilon), r^*(\epsilon))$ and $(y^{(1)*}(\epsilon), y^{(2)*}(\epsilon), s^*(\epsilon))$ defined in (9-14) are feasible solutions of problems $LP(\triangle b, \triangle c, \epsilon)$ and $LD(\triangle b, \triangle c, \epsilon)$, respectively. On the other hand, (15) proves that $s^*(\epsilon)^T x^{(2)*}(\epsilon) = 0$ holds. Moreover, the restrictions $\sigma(y^{(2)}) \subseteq \mathcal{B}_C^y$ and $\sigma(r) \subseteq \mathcal{N}_C^r$ guarantee the validity of $r^*(\epsilon)^T y^{(2)*}(\epsilon) = 0$ that proves the complementarity of $y^{(2)*}(\epsilon)$ and $r^*(\epsilon)$. Thus, $(\epsilon_\ell, \epsilon_u) \subseteq \text{int}(\Upsilon_L^G(\triangle b, \triangle c))$.

We also need to prove that $\text{int}(\Upsilon_L^G(\triangle b, \triangle c)) \subseteq (\epsilon_\ell, \epsilon_u)$. Let us assume to the contrary that $\bar{\epsilon} \in \text{int}(\Upsilon_L^G(\triangle b, \triangle c))$ but $\bar{\epsilon} \notin (\epsilon_\ell, \epsilon_u)$. Without loss of generality, one may consider that $\bar{\epsilon} < \epsilon_\ell$. By using Lemma 1, we have that every $\epsilon \in [\bar{\epsilon}, \epsilon_\ell)$ belongs to $\Upsilon_L^G(\triangle b, \triangle c)$ that contradicts the optimality of ϵ_ℓ in auxiliary problem (7). Thus, $\text{int}(\Upsilon_L^G(\triangle b, \triangle c)) \subseteq (\epsilon_\ell, \epsilon_u)$ that completes the proof. □

As in the proof of Theorem 4, if $\epsilon_\ell = \epsilon_u = 0$, then there is no possibility to perturb the RHS and OFC data of problem $LP(\triangle b, \triangle c, \epsilon)$ in the perturbing directions $\triangle b$ and $\triangle c$ while maintaining the GISS property of this problem. In this case, the GISS interval $\Upsilon_L^G(\triangle b, \triangle c)$ is the singleton $\{0\}$. On the other hand, if the auxiliary problem (7) is unbounded, then the left end of the GISS interval $\Upsilon_L^G(\triangle b, \triangle c)$ is $-\infty$. Analogous discussion is valid for the right end point ϵ_u, using the auxiliary problem (8).

The following lemma presents closedness conditions of the GISS interval $\Upsilon_L^G(\triangle b, \triangle c)$.

Lemma 2. *Let ϵ_ℓ and ϵ_u be as given in Theorem 4. Then $\epsilon_\ell \in \Upsilon_L^G(\triangle b, \triangle c)$ (or $\epsilon_u \in \Upsilon_L^G(\triangle b, \triangle c)$) if $\sigma(x^{(2)*}(\epsilon_\ell)) = P$ (or $\sigma(x^{(2)*}(\epsilon_u)) = P$).*

Proof. If $\epsilon_\ell = 0$ (or $\epsilon_u = 0$) then the GISS interval $\Upsilon_L^G(\triangle b, \triangle c)$ is at least half-closed. Let us consider that $\epsilon_\ell < 0$ and $(\epsilon_\ell, x^{(1)*}(\epsilon_\ell), x^{(2)*}(\epsilon_\ell), r^*(\epsilon_\ell), y^{(1)*}(\epsilon_\ell), y^{(2)*}(\epsilon_\ell), s^*(\epsilon_\ell))$ be an optimal solution of the auxiliary problem (7). Note that $\sigma(x^{(2)*}(\epsilon_\ell)) \subseteq P$ and $\sigma(r^*(\epsilon_\ell)) \subseteq \sigma(r^*)$ holds in the feasible solution set of this problem. However, it may exists a situation that $\sigma(x^{(2)*}(\epsilon_\ell)) = P$ but $\sigma(r^*(\epsilon_\ell)) \subset \sigma(r^*)$. In this case, $\epsilon_\ell \in \Upsilon_L^G(\triangle b, \triangle c)$. Analogous discussion is valid for ϵ_u that completes the proof. □

Remark 4. Observe that when the problem LP is in standard form i.e., when $n_1 = m_2 = 0$, we have $A = A_{12},\ b = b^{(1)},\ c = c^{(2)},\ \triangle b = \triangle b^{(1)},\ \triangle c = \triangle c^{(2)},\ x = x^{(2)}$ and $y = y^{(1)}$. In this case, the auxiliary LO problems (7) and (8) reduce to the problems that were presented in [5] for identifying the ISS interval $\Upsilon_L(\triangle b, \triangle c)$. It means that the general support set sensitivity analysis coincides with the standard support set sensitivity analysis, when the primal problem LP is in standard form.

Remark 5. When $n_1 = m_1 = 0$, then problem LP is in canonical form. In this case, $A = A_{22},\ b = b^{(2)},\ c = c^{(2)},\ \triangle b = \triangle b^{(2)},\ \triangle c = \triangle c^{(2)},\ x = x^{(2)}$ and $y = y^{(2)}$. In this case, two auxiliary LO problems (7) and (8) reduce to the following problems, respectively:

$$\begin{aligned}
\epsilon_\ell &= \min\{\epsilon | (x, r, y, s, \epsilon) \in \Phi_C,\ \sigma(y) \subseteq \mathcal{B}_C^y,\ \sigma(r) \subseteq \mathcal{N}_C^r,\ \sigma(x) \subseteq P,\ \sigma(s) \subseteq Z\},\\
\epsilon_u &= \max\{\epsilon | (x, r, y, s, \epsilon) \in \Phi_C,\ \sigma(y) \subseteq \mathcal{B}_C^y,\ \sigma(r) \subseteq \mathcal{N}_C^r,\ \sigma(x) \subseteq P,\ \sigma(s) \subseteq Z\},
\end{aligned}$$

where $\pi_C = (\mathcal{B}_C^y, \mathcal{N}_C^r)$ is the constraints optimal partition.

Let us specialize the method for the cases when either $\triangle b$ or $\triangle c$ is a zero vector.

- **Case** $\triangle c = 0$**:** In this case, for any $\epsilon \in \Upsilon_L^G(\triangle b, 0)$, the given dual optimal solution (y^*, s^*) is still a feasible (optimal) solution of problem $LD(\triangle b, 0, \epsilon)$, for any $\epsilon \in \Upsilon_G(\triangle b, 0)$. Thus, it is enough to find the range of parameter variation, where the problem $LP(\triangle b, 0, \epsilon)$ has a feasible solution $(x^{(1)}(\epsilon), x^{(2)}(\epsilon), r(\epsilon))$ with $\sigma(x^{(2)}(\epsilon)) = P$ and $\sigma(r(\epsilon)) \subseteq \mathcal{N}_C^r$. These two relations guarantee the optimality of $(x^{(1)}(\epsilon), x^{(2)}(\epsilon), r(\epsilon))$. The following corollary summarizes these discussions.

Corollary 3. *Let $(x^{(1)*}, x^{(2)*}, r^*)$ be a primal optimal solution of problems LP, where $\sigma(x^{(2)*}) = P$. Further, let $\pi_C = (\mathcal{B}_C^y, \mathcal{N}_C^r)$ be the constraints optimal partition of problem LP. Then, ϵ_ℓ and ϵ_u, the end points of $\overline{\Upsilon}_L^G(\triangle b, 0)$ can be determined by solving the following auxiliary LO problems, respectively:*

$$\epsilon_\ell = \min\{\epsilon | (x^{(1)}, x^{(2)}, r, \epsilon) \in \Phi_{LP},\ \sigma(r) \subseteq \mathcal{N}_C^r,\ \sigma(x^{(2)}) \subseteq P\}, \tag{16}$$
$$\epsilon_u = \max\{\epsilon | (x^{(1)}, x^{(2)}, r, \epsilon) \in \Phi_{LP},\ \sigma(r) \subseteq \mathcal{N}_C^r,\ \sigma(x^{(2)}) \subseteq P\}. \tag{17}$$

• **Case $\triangle b = 0$:** In this case, for any $\epsilon \in \Upsilon_L^G(0, \triangle c)$, the given primal optimal solution $(x^{(1)*}, x^{(2)*}, r^*)$ with $\sigma(x^{(2)*}) = P$ is still a feasible (optimal) solution of problem $LP(0, \triangle c, \epsilon)$. Thus, we only need to find the range of parameter variation, where the dual problem $LD(0, \triangle b, \epsilon)$ has a feasible solution $(y^{(1)}, y^{(2)}, s)$ with $\sigma(s) \subseteq Z$ and $\sigma(y^{(2)}) \subseteq \mathcal{B}_C^y$. The following corollary talks about this specialization.

Corollary 4. *Let $(x^{(1)*}, x^{(2)*}, r^*)$ be a primal optimal solution of problems LP, where $\sigma(x^{(2)*}) = P$. Further, let $\pi_C = (\mathcal{B}_C^y, \mathcal{N}_C^r)$ be the constraints optimal partition of problem LP. Then, ϵ_ℓ and ϵ_u, the end points of $\overline{\Upsilon}_L^G(0, \triangle c)$ can be determined by solving the following auxiliary LO problems, respectively:*

$$\epsilon_\ell = \min\{\epsilon | (y^{(1)}, y^{(2)}, s, \epsilon) \in \Phi_{LD},\ \sigma(y^{(2)}) \subseteq \mathcal{B}_C^y,\ \sigma(s) \subseteq Z\}, \tag{18}$$
$$\epsilon_u = \max\{\epsilon | (y^{(1)}, y^{(2)}, s, \epsilon) \in \Phi_{LD},\ \sigma(y^{(2)}) \subseteq \mathcal{B}_C^y,\ \sigma(s) \subseteq Z\}. \tag{19}$$

Remark 6. Observe that the two auxiliary LO problems (18) and (19) in Corollary 4 to identify ϵ_ℓ and ϵ_u are the same problems that were presented in Theorem 2 to identify the actual OPI interval when $\triangle b = 0$. Thus, by considering Lemma 2, the interior of the GISS interval $\Upsilon_G(0, \triangle c)$ coincides with the actual OPI interval in this case.

The following theorem talks about a trivial relationship between the GISS intervals in simultaneous and nonsimultaneous cases. It is a direct consequence of the relation between the feasible solution sets of auxiliary LO problems (7)-(19).

Theorem 5. *$\Upsilon_L^G(\triangle b, \triangle c) = \Upsilon_L^G(\triangle b, 0) \cap \Upsilon_L^G(0, \triangle c)$.*

3.2.2. *Identifying the GIACS intervals.* The following theorem presents two auxiliary LO problems that enable us to identify γ_ℓ and γ_u, i.e., the end points of $\overline{\Gamma}_L^G(\triangle b, \triangle c)$, where $\overline{\Gamma}_L^G(\triangle b, \triangle c)$ denotes the closure of the GIACS interval $\Gamma_L^G(\triangle b, \triangle c)$. The proof is analogous to the proof of Theorem 4 and it is omitted.

Theorem 6. *Let $(x^{(1)*}, x^{(2)*}, r^*)$ be a primal optimal solution of problem LP, where $\sigma(r^*) = \widehat{P}$. Moreover, let $\pi_V = (\mathcal{B}_V^x, \mathcal{N}_V^s)$ be the variables optimal partition of this problem. Then, γ_ℓ and γ_u, i.e., the end points of $\overline{\Gamma}_L^G(\triangle b, \triangle c)$, can be obtained by* *solving the following auxiliary LO problems, respectively:*

$$\begin{aligned}\gamma_\ell &= \min\{\epsilon|(x^{(1)},x^{(2)},r,y^{(1)},y^{(2)},s,\epsilon)\in\Phi,\\ &\qquad \sigma(x^{(2)})\subseteq\mathcal{B}_V^x,\ \sigma(s)\subseteq\mathcal{N}_V^s,\ \sigma(r)\subseteq\widehat{P},\ \sigma(y^{(2)})\subseteq\widehat{Z}\}, \quad (20)\\ \gamma_u &= \max\{\epsilon|(x^{(1)},x^{(2)},r,y^{(1)},y^{(2)},s,\epsilon)\in\Phi,\\ &\qquad \sigma(x^{(2)})\subseteq\mathcal{B}_V^x,\ \sigma(s)\subseteq\mathcal{N}_V^s,\ \sigma(r)\subseteq\widehat{P},\ \sigma(y^{(2)})\subseteq\widehat{Z}\}. \quad (21)\end{aligned}$$

In solving problems (20) and (21), if $\gamma_\ell=\gamma_u=0$, then there is no possibility to perturb the RHS and OFC data of problem $LP(\triangle b,\triangle c,\epsilon)$ in perturbing directions $\triangle b$ and $\triangle c$ while maintaining the GIACS property of this problem. In this case, the GIACS interval $\Gamma_L^G(\triangle b,\triangle c)$ is the singleton $\{0\}$. On the other hand, if the auxiliary problem (20) is unbounded, then the left end of the GIACS interval $\Gamma_L^G(\triangle b,\triangle c)$ is $-\infty$. Analogous discussion is valid for the right end point γ_u.

Analogous to Lemma 2, one can easily establish the following conditions for the closedness of the GIACS interval $\Gamma_L^G(\triangle b,\triangle c)$. The proof goes similarly and it is omitted.

Lemma 3. *Let γ_ℓ and γ_u be as defined in Theorem 6. Then $\gamma_\ell\in\Gamma_L^G(\triangle b,\triangle c)$ (or $\gamma_u\in\Gamma_L^G(\triangle b,\triangle c)$) if $\sigma(r^*(\gamma_\ell))=\widehat{P}$ (or $\sigma(r^*(\gamma_u))=\widehat{P}$).*

Remark 7. Observe that the GIACS sensitivity analysis for problem LP, when it is in standard form, does not make sense. However, one may apply the presented method of Theorem 6 to the dual problem LD to find the range of parameter variation, where the slack vector s has the same support set in dual optimal solutions. In this case, the results of the GIACS sensitivity analysis are specialized to the results that were obtained in [3].

Remark 8. When the problem LP is in canonical form, then the two auxiliary LO problems (18) and (19) reduce to the following problems, respectively:

$$\begin{aligned}\gamma_\ell &= \min\{\epsilon|(x,r,y,s,\epsilon)\in\Phi_C,\ \sigma(x)\subseteq\mathcal{B}_V^x,\ \sigma(s)\subseteq\mathcal{N}_V^s,\ \sigma(r)\subseteq\widehat{P},\ \sigma(y)\subseteq\widehat{Z}\},\\ \gamma_u &= \max\{\epsilon|(x,r,y,s,\epsilon)\in\Phi_C,\ \sigma(x)\subseteq\mathcal{B}_V^x,\ \sigma(s)\subseteq\mathcal{N}_V^s,\ \sigma(r)\subseteq\widehat{P},\ \sigma(y)\subseteq\widehat{Z}\},\end{aligned}$$

where $\pi_V=(\mathcal{B}_V^x,\mathcal{N}_V^s)$ is the variables optimal partition.

One can establish the following corollaries to specialize the method introduced in Theorem 6 to identify the GIACS intervals for the cases when either $\triangle b$ or $\triangle c$ is a zero vector. The reasonings are similar to Corollaries 3 and 4 and they are omitted.

Corollary 5. *Let $(x^{(1)*},x^{(2)*},r^*)$ be a primal optimal solution of problem LP, where $\sigma(r^*)=\widehat{P}$. Further, let $\pi_V=(\mathcal{B}_V^x,\mathcal{N}_V^s)$ be the variables optimal partition of this problem. Then, γ_ℓ and γ_u the end points of $\overline{\Gamma}_L^G(\triangle b,0)$ can be obtained by solving the following two auxiliary LO problems, respectively:*

$$\begin{aligned}\gamma_\ell &= \min\{\epsilon|(x^{(1)},x^{(2)},r,\epsilon)\in\Phi_{LP},\ \sigma(x^{(2)})\subseteq\mathcal{B}_V^x,\ \sigma(r)\subseteq\widehat{P}\},\\ \gamma_u &= \max\{\epsilon|(x^{(1)},x^{(2)},r,\epsilon)\in\Phi_{LP},\ \sigma(x^{(2)})\subseteq\mathcal{B}_V^x,\ \sigma(r)\subseteq\widehat{P}\}.\end{aligned}$$

Corollary 6. *Let $(x^{(1)*},x^{(2)*},r^*,y^{(1)*},y^{(2)*},s^*)$ be a primal-dual optimal solution of problems LP and LD, where $\sigma(r^*)=\widehat{P}$. Moreover, let $\pi_V=(\mathcal{B}_V^x,\mathcal{N}_V^s)$ be the variables optimal partition of this problem. Then, γ_ℓ andγ_u the end points* *of* $\overline{\Gamma}_L^G(0,\triangle c)$ *can be obtained by solving the following two auxiliary LO problems, respectively:*

$$\begin{aligned}\gamma_\ell &= \min\{\epsilon|(y^{(1)},y^{(2)},s,\epsilon)\in\Phi_{LD},\ \sigma(s)\subseteq\mathcal{N}_V^s,\ \sigma(y^{(2)})\subseteq\widehat{Z}\},\\ \gamma_u &= \max\{\epsilon|(y^{(1)},y^{(2)},s,\epsilon)\in\Phi_{LD},\ \sigma(s)\subseteq\mathcal{N}_V^s,\ \sigma(y^{(2)})\subseteq\widehat{Z}\}.\end{aligned}$$

Remark 9. Observe that the problems presented in Corollary 6 are the same problems that were presented in Theorem 2 to identify the actual OPI interval, when $\triangle b=0$. Thus, the GIACS interval $\Gamma_L^G(0,\triangle c)$ coincides with the actual OPI interval in this case.

The following theorem resembles Theorem 5, referring to a relationship between the GIACS intervals in simultaneous and nonsimultaneous cases. The reasoning is analogous and it is omitted.

Theorem 7. $\Gamma_L^G(\triangle b,\triangle c)=\Gamma_L^G(\triangle b,0)\cap\Gamma_L^G(0,\triangle c)$.

3.2.3. *Identifying the GIC intervals.* The following theorem presents computable methods to identify the end points of the closure of the GIC interval $\Theta_L^G(\triangle b,\triangle c)$.

Theorem 8. *Let* $(x^{(1)*},x^{(2)*},r^*,y^{(1)*},y^{(2)*},s^*)$ *be a primal-dual optimal solution of problems LP and LD, where* $\sigma(x^{(2)*})=P$ *and* $\sigma(r^*)=\widehat{P}$. *Further, let* $\pi_V=(\mathcal{B}_V^x,\mathcal{N}_V^s)$ *and* $\pi_C=(\mathcal{B}_C^y,\mathcal{N}_C^r)$ *be variables and constraints optimal partitions of the problem LP, respectively. Then,* θ_ℓ *and* θ_u*, i.e., the end points of* $\overline{\Theta}_L^G(\triangle b,\triangle c)$*, can be obtained by solving the following auxiliary LO problems, respectively:*

$$\begin{aligned}\theta_\ell &= \min\{\epsilon|(x^{(1)},x^{(2)},r,y^{(1)},y^{(2)},s,\epsilon)\in\Phi,\\ &\qquad \sigma(s)\subseteq\mathcal{N}_V^s,\ \sigma(y^{(2)})\subseteq\mathcal{B}_C^y,\sigma(x^{(2)})\subseteq P,\ \sigma(r)\subseteq\widehat{P}\},\\ \theta_u &= \max\{\epsilon|(x^{(1)},x^{(2)},r,y^{(1)},y^{(2)},s,\epsilon)\in\Phi,\\ &\qquad \sigma(s)\subseteq\mathcal{N}_V^s,\ \sigma(y^{(2)})\subseteq\mathcal{B}_C^y,\sigma(x^{(2)})\subseteq P,\ \sigma(r)\subseteq\widehat{P}\}.\end{aligned}$$

Remark 10. Observe that when the LP problem is in canonical form, then $m_1=n_1=0$. One may convert this canonical problem to standard form by inserting the reduced price vector r and applying the method that was introduced in [5]. In this case, the obtained ISS interval $\Upsilon_L(\triangle b,\triangle c)$ (for more details see [5]) coincides with the GIC interval $\Theta_L^G(\triangle b,\triangle c)$.

The following observation is straightforward and presents the relationships between the intervals $\Theta_L^G(\triangle b,\triangle c)$, $\Upsilon_L^G(\triangle b,\triangle c)$ and $\Gamma_L^G(\triangle b,\triangle c)$ when either $\triangle b$ or $\triangle c$ is a zero vector.

Remark 11. When $\triangle c=0$, then $\Theta_L^G(\triangle b,0)=\Upsilon_L^G(\triangle b,0)$ and when $\triangle b=0$, then $\Theta_L^G(0,\triangle c)=\Gamma_L^G(0,\triangle c)$.

4. **Illustrative Example.** Consider the following general version of the transportation problem. It provides a descriptive illustration of a possible practical example.

Example: Consider a transportation network with m stores and n destinations. Let the amount of a product in store i be a_i for $i=1,2,\ldots,m$ and the amount of the product that is needed in destination j be b_j for $j=1,2,\ldots,m$. Let x_{ij} are the amount of the product that should be transported from store i to destination j, that is not necessarily nonnegative. Moreover, let c_{ij} be the corresponding costs of transportation. We mention that in this general version of the transportation problem, the amount of available product in the stores is not equal to the amount of the product that is needed in destinations, i.e., $\sum_{i=1}^{n} a_i \neq \sum_{j=1}^{m} b_j$. Moreover, some of variables x_{ij} might be free variable. The possible negativity of this variable x_{ij} means that the product might be send back from destination j to store i. However, there is no allowance to transport any product between stores and between destinations themselves.

Let us go further with numerical data. Let $m = n = 2$ and x_{11} be a free variable. For consistency with the notation of this paper, let $x_{11} = x_1^{(1)}, x_{12} = x_1^{(2)}, x_{21} = x_2^{(2)}$ and $x_{22} = x_3^{(2)}$.Analogous substitutions are made for the coefficients c_{ij}. In this way, $x^{(1)} = (x_1^{(1)})$ and $x^{(2)^T} = (x_1^{(2)^T}, x_2^{(2)^T}, x_3^{(2)})^T \geq 0$. An instance of the primal problem might be given be as:

$$\begin{array}{llllll} \max & & -x_1^{(2)} & -2x_2^{(2)} & -3x_3^{(2)} & \\ \text{s.t.} & x_1^{(1)} & & +x_2^{(2)} & & = 16 \\ & & x_1^{(2)} & & +x_3^{(2)} & = 16 \\ & x_1^{(1)} & +x_1^{(2)} & & & \leq 20 \\ & & & x_2^{(2)} & +x_3^{(2)} & \leq 20 \\ & & x_1^{(2)}, & x_2^{(2)}, & x_3^{(2)} & \geq 0. \end{array} \tag{22}$$

The dual of (22) can be written as

$$\begin{array}{llllll} \min & 16y_1^{(1)} & +16y_2^{(1)} & +20y_1^{(2)} & +20y_2^{(2)} & \\ \text{s.t.} & y_1^{(1)} & & +y_1^{(2)} & & = 0 \\ & & y_2^{(1)} & +y_1^{(2)} & & \geq -1 \\ & y_1^{(1)} & & & +y_2^{(2)} & \geq -2 \\ & & y_2^{(1)} & & +y_2^{(2)} & \geq -3 \\ & & & y_1^{(2)}, & y_2^{(2)} & \geq 0. \end{array} \tag{23}$$

In problem (22), the variables index set is $\{1, 2, 3\}$ and the constraints index set is $\{1, 2\}$. Observe that dual problem (23) has a unique degenerate optimal solution $(y^{(1)*}, y^{(2)*}, s^*)$, where $y^{(1)*^T} = (-2, -3)$, $y^{(2)*^T} = (2, 0)$ and $s^{*T} = (0, 0, 0)$. Moreover, the primal problem (22) has two (nondegenerate) optimal basic solutions. Let $(\bar{x}^{(1)}, \bar{x}^{(2)}, \bar{r})$ and $(\tilde{x}^{(1)}, \tilde{x}^{(2)}, \tilde{r})$ denote these basic solutions, where $\bar{x}^{(1)} = 16$, $\bar{x}^{(2)} = (4, 0, 12)^T$, $\tilde{x}^{(1)} = 4$, $\tilde{x}^{(2)} = (16, 12, 0)^T$ and $\bar{r} = \tilde{r} = (0, 8)^T$. Thus, strictly complementary optimal solutions exist, e.g., $(x^{(1)*}, x^{(2)*}, r^*)$, where $x^{(1)*} = 10$, $x^{(2)*} = (10, 6, 6)^T$ and $r^{*T} = \bar{r}^* = (0, 8)^T$. Consequently, the variables and constraints optimal partitions are

$$\begin{aligned} \pi_V &= (\mathcal{B}_V^x, \mathcal{N}_V^r) = (\{1, 2, 3\}, \emptyset), \\ \pi_C &= (\mathcal{B}_C^y, \mathcal{N}_C^s) = (\{1\}, \{2\}). \end{aligned}$$

Let $\triangle b = (-1, 2, 1, -1)^T$ and $\triangle c = (0, -1, 0, 0)^T$ be the perturbation vectors. For these perturbation vectors, the OPI interval is the singleton $\{0\}$. Thus, $\epsilon = 0$ is a transition point of the optimal value function of this problem. It means that the actual OPI invariancy interval is the singleton $\{0\}$ too.

One may categorize the primal optimal solutions in two classes:

- **Nondegenerate primal optimal basic solution:** As we mentioned before, this problem has two nondegenerate primal optimal basic solution.

Let $(\bar{x}^{(1)}, \bar{x}^{(2)}, \bar{r})$ the given optimal solution. For this optimal solution, we have $P = \{1,3\}$ and $Z = \{2\}$. It is easy to verify that the GISS interval is $[0, 2)$. On the other hand, let $(\tilde{x}^{(1)}, \tilde{x}^{(2)}, \tilde{r})$ be the given basic optimal solution. For this optimal solution, we have $P = \{1,2\}$, $Z = \{3\}$. Therefore, the GISS interval is $(-8, 0]$. Observe that these GISS intervals include the actual OPI interval (the singleton $\{0\}$) and the right and the left OPI intervals, respectively.

- **Strictly complementary primal-dual optimal solution:** Let $(x^{(1)*}, x^{(2)*}, r^*, y^{(1)*}, y^{(2)*}, s^*)$ be the given primal-dual strictly complementary optimal solution. In this case, the GISS interval is the singleton $\{0\}$ and this interval and the actual OPI interval coincide.

For the case of degenerate optimal basic solution, let us consider the dual problem (23), that has a unique degenerate optimal solution. For this problem, the (dual) GISS interval is $\{0\}$. Because of the symmetry between general primal and dual problems, the GISS intervals for the general primal LO problem coincide the GIACS interval for the general dual LO problem and vice versa. Thus, the GIACS interval for problem (22) is $\{0\}$, as well.

5. **Conclusions.** In this paper we developed the support set sensitivity analysis for general LO problem. We presented axillary LO problems that enable us to identify the associated intervals. All these auxiliary LO problems can be solved in polynomial time using an IPM. Since the support set invariancy sensitivity analysis has been studied in CQO [3, 6], these methods can be generalized for CQO.

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