# McMaster University

## Advanced Optimization Laboratory

**Title:**

Berge Sorting


**Authors:**

Antoine Deza and William Hua



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# Berge Sorting

*Dedicated to Professor Masakazu Kojima on the occasion of his 60th birthday*

Antoine Deza William Hua

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**Abstract**

In 1966, Claude Berge proposed the following sorting problem. Given a string of $n$ alternating white and black pegs on a one-dimensional board consisting of an unlimited number of empty holes, rearrange the pegs into a string consisting of $\lceil \frac{n}{2} \rceil$ white pegs followed immediately by $\lfloor \frac{n}{2} \rfloor$ black pegs (or vice versa) using only moves which take 2 adjacent pegs to 2 vacant adjacent holes. Avis and Deza proved that the alternating string can be sorted in $\lceil \frac{n}{2} \rceil$ such *Berge* 2*-moves* for $n \geq 5$. Extending Berge's original problem, we consider the same sorting problem using *Berge* $k$*-moves*, i.e., moves which take $k$ adjacent pegs to $k$ vacant adjacent holes. We prove that the alternating string can be sorted in $\lceil \frac{n}{2} \rceil$ Berge 3-moves for $n \not\equiv 0 \pmod 4$ and in $\lceil \frac{n}{2} \rceil + 1$ Berge 3-moves for $n \equiv 0 \pmod 4$, for $n \geq 5$. In general, we conjecture that, for any $k$ and large enough $n$, the alternating string can be sorted in $\lceil \frac{n}{2} \rceil$ Berge $k$-moves. This estimate is tight as $\lceil \frac{n}{2} \rceil$ is a lower bound for the minimum number of required Berge $k$-moves for $k \geq 2$ and $n \geq 5$.

## 1 Introduction

In a column that appeared in the Revue Française de Recherche Opérationnelle in 1966, entitled *Problèmes plaisans et délectables* in homage to the 17th century work of Bachet [2], Claude Berge [3] proposed the following sorting problem:

> For $n \geq 5$, given a string of $n$ alternating white and black pegs on a one-dimensional board consisting of an unlimited number of empty holes, we are required to rearrange the pegs into a string consisting of $\lceil \frac{n}{2} \rceil$ white pegs followed immediately by $\lfloor \frac{n}{2} \rfloor$ black pegs (or vice versa) using only moves which take 2 adjacent pegs to 2 vacant adjacent holes. Berge noted that the minimum number of moves required is 3 for $n = 5$ and 6, and 4 for $n = 7$. See Figure 1 for a sorting of 5 pegs in 3 moves.

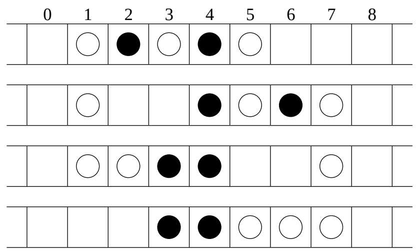


Figure 1: Sorting 5 pegs in 3 moves

Avis and Deza [1] provided a solution in $\lceil \frac{n}{2} \rceil$ *Berge* 2-*moves* for $n \geq 5$. Extending Berge's original problem, we consider the same sorting question using only Berge $k$-moves, i.e., moves which take $k$ adjacent pegs to $k$ vacant adjacent holes. We provide a solution in $\lceil \frac{n}{2} \rceil$ Berge 3-moves for $n \not\equiv 0 \pmod 4$ and in $\lceil \frac{n}{2} \rceil + 1$ Berge 3-moves for $n \equiv 0 \pmod 4$ and $n \geq 5$. The authors generated minimal solutions by computer for a large number of $k$ and $n$ which turned out all be equal to $\lceil \frac{n}{2} \rceil$ except for the few first small values of $n$. Note that, for $k \geq 2$, $\lceil \frac{n}{2} \rceil$ is a lower bound for the minimum number of required Berge $k$-moves, see Section 3.1. To the best of our knowledge, this property was not noticed earlier. We conjecture that for any $k$ and large enough $n$, the alternating string can be rearranged into a string consisting of $\lceil \frac{n}{2} \rceil$ white pegs followed immediately by $\lfloor \frac{n}{2} \rfloor$ black pegs (or vice versa) by only $\lceil \frac{n}{2} \rceil$ moves which take $k$ adjacent pegs to $k$ vacant adjacent holes.

## 2 Notation

We follow and adapt the notation used in [1, 3]. The starting game board consists of $n$ alternating white and black pegs sitting in the positions 1 through $n$. A single Berge $k$-move will be denoted as $\{\ j\ i\ \}$, in which case, the pegs in the positions $i, i+1, \ldots, i+k-1$ are moved to the vacant holes $j, j+1, \ldots, j+k-1$. Successive moves are concatenated as $\{\ j\ i\ \} \cup \{\ l\ k\ \}$, which means perform $\{\ j\ i\ \}$ followed by $\{\ l\ k\ \}$. Often, a move fills an empty hole created as an effect of the previous move, and the resulting notation $\{\ j\ k\ \} \cup \{\ k\ i\ \}$ is abbreviated as $\{\ j\ k\ i\ \}$. This can be extended to more than two such moves as well. $\mathcal{S}_{n,k}$ denotes a solution for $n$ pegs by Berge $k$-moves and $h(n,k)$ denotes the minimum number of required $k$-moves, i.e., the length of a shortest solution. For example, with this notation, possible solutions corresponding to the values $h(5,2) = h(6,2) = 3$ and $h(7,2) = 4$ given by Berge [3] are illustrated in Table 1.

Table 1: First solutions using Berge 2-moves

$$\begin{aligned} \mathcal{S}_{5,2} &= \{\ 6\ \ 2\ \ 5\ \ 1\ \} \\ \mathcal{S}_{6,2} &= \{\ 7\ \ 4\ \ 1\ \} \cup \{\ 9\ \ 3\ \} \\ \mathcal{S}_{7,2} &= \{\ 8\ \ 2\ \ 5\ \ 8\ \ 1\ \} \end{aligned}$$

# 3 Main Results

## 3.1 Minimum number of required Berge $k$-moves

Let $\mathcal{D}_{n,k}(i)$ denote the *disorder*, i.e., the number of pegs whose right neighbour is not a peg of the same colour after the $i$-th Berge $k$-move. One can easily check that $|\mathcal{D}_{n,k}(i)-\mathcal{D}_{n,k}(i+1)| \leq 2$. A move such that $\mathcal{D}_{n,k}(i)-\mathcal{D}_{n,k}(i+1) = 2$ (resp. 1 and 0) is called *optimal* (resp. *suboptimal* and *neutral*).

**Lemma 3.1.** *For $k \geq 1$ and $n \geq 3$, at least $\lfloor \frac{n}{2} \rfloor$ Berge $k$-moves are required to sort a string of $n$ alternating white and black pegs. In other words, $h(n,k) \geq \lfloor \frac{n}{2} \rfloor$ for $k \geq 1$ and $n \geq 3$.*

*Proof.* *The disorder of the initial board is $\mathcal{D}_{n,k}(0) = n$ and the disorder of the sorted string is $\mathcal{D}_{n,k}(h(n,k)) = 2$. Since the first move cannot be optimal, i.e., $\mathcal{D}_{n,k}(0) - \mathcal{D}_{n,k}(1) \leq 1$, and the following moves satisfy $\mathcal{D}_{n,k}(i) - \mathcal{D}_{n,k}(i+1) \leq 2$, we have $h(n,k) \geq \lfloor \frac{n}{2} \rfloor$.* □

Table 2: Sorting $n$ pegs in $\lfloor \frac{n}{2} \rfloor$ Berge 1-moves for $n \equiv 3 \pmod 4$

$$\begin{aligned} \mathcal{S}_{3,1} &= \{\ 4\ \ 1\ \} \\ \mathcal{S}_{7,1} &= \{\ 8\ \ 3\ \ 6\ \ 1\ \} \\ \mathcal{S}_{11,1} &= \{\ 12\ \ 3\ \ 10\ \ 5\ \ 8\ \ 1\ \} \\ \mathcal{S}_{15,1} &= \{\ 16\ \ 3\ \ 14\ \ 5\ \ 12\ \ 7\ \ 10\ \ 1\ \} \\ \mathcal{S}_{4i+3,1} &= \{\ 4i{+}4\ \ 3\ \ 4i{+}2\ \ 5\ \ 4i\ \ 7\ \ 4i{-}2\ \ 9\ \ \ldots\ \ 2i{+}4\ \ 1\ \} \end{aligned}$$

Lemma 3.1 is tight because, for $k = 1$, we have $h(n,1) = \lfloor \frac{n}{2} \rfloor$ for $n \equiv 3 \pmod 4$, see Table 2. Solutions in $\lceil \frac{n}{2} \rceil$ Berge 1-moves for $n \not\equiv 3 \pmod 4$ are very similar to the ones in $\lfloor \frac{n}{2} \rfloor$ 1-moves for $n \equiv 3 \pmod 4$. Avis and Deza noticed in [1] that $h(n,2) \geq \lceil \frac{n}{2} \rceil$ for $n \geq 5$. For $k \geq 2$, Lemma 3.1 can be strengthen to the following lemma.

**Lemma 3.2.** *For $k \geq 2$ and $n \geq 5$, at least $\lceil \frac{n}{2} \rceil$ Berge $k$-moves are required to sort a string of $n$ alternating white and black pegs. In other words, $h(n,k) \geq \lceil \frac{n}{2} \rceil$ for $k \geq 2$ and $n \geq 5$.*

*Proof.* *As Lemma 3.1 and 3.2 are equivalent for even $n$, let us assume that, for odd $n \geq 5$, we have a solution in $\lfloor \frac{n}{2} \rfloor$ Berge $k$-moves. It implies that, after the first suboptimal move, all the following moves are optimal. We derive a contradiction for $k = 3$ and the same argument can be used for any $k \geq 2$. Since $n$ is odd, the initial board is something* *like ○ ● ○ ● ○ ● ○ ● ○ ● ○ where ○ and ● represent white and black pegs. By symmetry, we can assume the first move is to the right. This first suboptimal move has to take* 3 *pegs from the interior of the string to the position* $n+1$*. For example, with* $n = 11$*, the board after the first move is something like ○ ● − − − ● ○ ● ○ ● ○ ○ ●○. The next move must fill the vacancy with a ● ⋆ ● triple, where ⋆ is any colour, but additionally the ● ⋆ ● triple must have been taken from between two white pegs to maintain optimality. Similarly, the subsequent moves must alternate between optimal fillings of ● − − − ● and ○ − − − ○ vacancies. Consider the last* 4 *(or* $k+1$ *in general) pegs, ○ ○ ●○, after the first suboptimal move: As the last triple, ○ ● ○, or the triple before, ○ ○ ●, do not correspond to an optimal filling, the black (or white) peg in the last 2 positions cannot be sorted by optimal moves.* □

### 3.2 Optimal solutions for sorting by Berge $k$-moves

We first recall that a solution for sorting the alternating string in $\lceil \frac{n}{2} \rceil$ Berge 2-moves for $n \geq 5$ was given in [1].

**Proposition 3.3.** [1] *For* $n \geq 5$*, a string of* $n$ *alternating white and black pegs can be sorted in* $\lceil \frac{n}{2} \rceil$ *Berge* 2*-moves. In other words,* $h(n, 2) = \lceil \frac{n}{2} \rceil$ *for* $n \geq 5$*.*

Considering the case $k = 3$, we prove that $h(n, 3) = \lceil \frac{n}{2} \rceil$ for $n \not\equiv 0 \pmod 4$ and, while computer calculations and preliminary attempts strongly indicated that the same holds for $n \equiv 0 \pmod 4$ and $n \geq 20$, so far we could only exhibit a solution in $\lceil \frac{n}{2} \rceil + 1$ Berge 3-moves for $n \equiv 0 \pmod 4$ and $n \geq 8$.

**Proposition 3.4.** *For* $n \geq 5$*, a string of* $n$ *alternating white and black pegs can be sorted in* $\lceil \frac{n}{2} \rceil$ *Berge* 3*-moves for* $n \not\equiv 0 \pmod 4$ *and in* $\lceil \frac{n}{2} \rceil + 1$ *Berge* 3*-moves for* $n \equiv 0 \pmod 4$*. In other words, for* $n \geq 5$*,* $h(n, 3) = \lceil \frac{n}{2} \rceil$ *for* $n \not\equiv 0 \pmod 4$ *and* $\lceil \frac{n}{2} \rceil \leq h(n, 3) \leq \lceil \frac{n}{2} \rceil + 1$ *for* $n \equiv 0 \pmod 4$*.*

*Proof. See Section 3.3 for a description of the solutions* $\mathcal{S}_{n,3}$*.* □

Propositions 3.3 and 3.4 lead to the following conjecture.

**Conjecture 3.5.** *For* $k \geq 2$ *and* $n \geq 2k + 11$*, a string of* $n$ *alternating white and black pegs can be sorted in* $\lceil \frac{n}{2} \rceil$ *Berge* $k$*-moves. In other words,* $h(n, k) = \lceil \frac{n}{2} \rceil$ *for* $k \geq 2$ *and* $n \geq 2k + 11$*.*

To substantiate Conjecture 3.5, the authors calculated the values of $h(n, k)$ by computer for $k \leq 14$ and $n \leq 50$ and, for these preliminary computations, did not find any counterexample. See Table 9, which gives the values of $h(n, k) - \lceil \frac{n}{2} \rceil$ for $k \leq 14$ and $n \leq 50$. Note that the alternating string obviously cannot be sorted by any number of $k$-moves for $n \leq k + 1$. The more conservative conjecture consisting in replacing "$n \geq 2k+11$" by "$n \geq \binom{k+2}{2} + 7$" is also consistent with the computations reported in Table 9. See [4] for detailed and updated computational results.

### 3.3 Proof of Proposition 3.4

We exhibit solutions $\mathcal{S}_{n,3}$ in $\lceil \frac{n}{2} \rceil$ moves for $n \not\equiv 0 \pmod 4$ and in $\lceil \frac{n}{2} \rceil + 1$ moves for $n \equiv 0 \pmod 4$.

#### 3.3.1 Case $n \equiv 1 \pmod 4$

We have $\mathcal{S}_{5,3} = \{\ 6\ 2\ 5\ 1\ \}$ and $\mathcal{S}_{n,3}$ can be constructed inductively as follows. Let $n = 4i+1 \geq 9$ and assume we have a solution $\mathcal{S}_{4i-3,3}$ taking $\lceil \frac{4i-3}{2} \rceil$ moves. First ignore the 4 pegs in positions $1, 2, 2i+3$ and $2i+4$ and sort the remaining $4i-3$ pegs using the solution $\mathcal{S}_{4i-3,3}$. Then complete the solution $\mathcal{S}_{4i+1,3}$ by the 2 moves $\{\ 3\ \ 2i+4\ \ 1\ \}$. The solution $\mathcal{S}_{4i+1,3}$ takes $\lceil \frac{4i-3}{2} \rceil + 2 = \lceil \frac{n}{2} \rceil$ moves. Note that the solution $\mathcal{S}_{4i-3,3}$ can be performed while ignoring the 4 pegs in positions $1, 2, 2i+3$ and $2i+4$ because these pegs are not moved as, by induction, the solution $S_{4i+1,3}$ does not include among its entries any of $-1$, $0$, $2i+1$, or $2i+2$ in the first $2i-1$ moves for $i \geq 1$. More precisely, with $\mathcal{S}^j_{n,3}$ denoting the $j$-th entry of the solution $\mathcal{S}_{n,3}$, we have:

$$\mathcal{S}^j_{4i+1,3} = \begin{cases} \mathcal{S}^j_{4i-3,3} + 2 & \text{for } 1 \leq \mathcal{S}^j_{4i-3,3} \leq 2i-2 \\ \mathcal{S}^j_{4i-3,3} + 4 & \text{for } 2i+1 \leq \mathcal{S}^j_{4i-3,3} \end{cases} .$$

See Table 3 for the first solutions $\mathcal{S}_{n,3}$ for $n = 5, 9, 13$ and 17 and Figure 2 illustrating the induction from $\mathcal{S}_{5,3}$ to $\mathcal{S}_{9,3}$.

Table 3: First solutions for sorting $n$ pegs in $\lceil \frac{n}{2} \rceil$ Berge 3-moves for $n \equiv 1 \pmod 4$

$$\begin{array}{rcl} \mathcal{S}_{5,3} & = & \{\ 6\ \ 2\ \ 5\ \ 1\ \} \\ \mathcal{S}_{9,3} & = & \{\ 10\ \ 4\ \ 9\ \ 3\ \ 8\ \ 1\ \} \\ \mathcal{S}_{13,3} & = & \{\ 14\ \ 6\ \ 13\ \ 5\ \ 12\ \ 3\ \ 10\ \ 1\ \} \\ \mathcal{S}_{17,3} & = & \{\ 18\ \ 8\ \ 17\ \ 7\ \ 16\ \ 5\ \ 14\ \ 3\ \ 12\ \ 1\ \} \end{array}$$

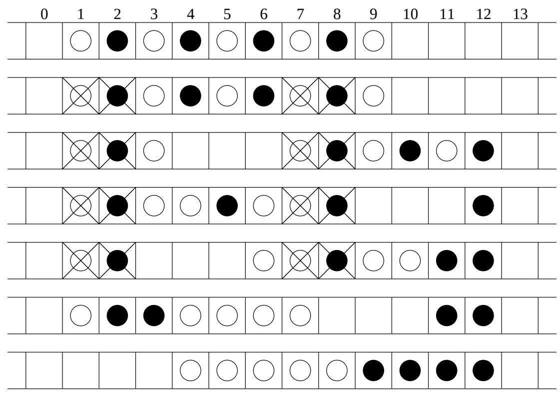


Figure 2: Sorting 9 pegs using the solution for 5

### 3.3.2 Case $n \equiv 2 \pmod 4$

We have $\mathcal{S}_{6,3} = \{\ 7\ 2\ 6\ 1\ \}$ and $\mathcal{S}_{n,3}$ can be constructed inductively as follows. Let $n = 4i+2 \geq 10$ and assume we have a solution $\mathcal{S}_{4i-2,3}$ taking $\lceil \frac{4i-2}{2} \rceil$ moves. First ignore the 4 pegs in positions $1, 2, 2i+3$ and $2i+4$ and sort the remaining $4i-2$ pegs using the solution $\mathcal{S}_{4i-2,3}$. Then complete the solution $\mathcal{S}_{4i+2,3}$ by the 2 moves $\{\ 3\ \ 2i+4\ \ 1\ \}$. The solution $\mathcal{S}_{4i+2,3}$ takes $\lceil \frac{4i-2}{2} \rceil + 2 = \lceil \frac{n}{2} \rceil$ moves. Note that the solution $\mathcal{S}_{4i-2,3}$ can be performed while ignoring the 4 pegs in positions $1, 2, 2i+3$ and $2i+4$ because, by an argument similar to the one used in Section 3.3.1, these pegs are not moved. See Table 4 for the first solutions $\mathcal{S}_{n,3}$ for $n = 6, 10, 14$ and 18.

Table 4: First solutions for sorting $n$ pegs in $\lceil \frac{n}{2} \rceil$ Berge 3-moves for $n \equiv 2 \pmod 4$

$$
\begin{aligned}
\mathcal{S}_{6,3} &= \{\ 7\ \ 2\ \ 6\ \ 1\ \} \\
\mathcal{S}_{10,3} &= \{\ 11\ \ 4\ \ 10\ \ 3\ \ 8\ \ 1\ \} \\
\mathcal{S}_{14,3} &= \{\ 15\ \ 6\ \ 14\ \ 5\ \ 12\ \ 3\ \ 10\ \ 1\ \} \\
\mathcal{S}_{18,3} &= \{\ 19\ \ 8\ \ 18\ \ 7\ \ 16\ \ 5\ \ 14\ \ 3\ \ 12\ \ 1\ \}
\end{aligned}
$$

The following lemma can be easily checked by induction.

**Lemma 3.6.**

*(i)* *For $n \equiv 2 \pmod 4$, the solutions $\mathcal{S}_{n,3}$ shift the string three spaces to the right overall.*

*(ii)* *For $n \equiv 2 \pmod 4$, the solutions $\mathcal{S}_{n,3}$ place the $\lceil \frac{n}{2} \rceil$ white pegs to the left of the $\lfloor \frac{n}{2} \rfloor$ black pegs.*

#### 3.3.3 Case $n \equiv 3 \pmod 4$

We have $\mathcal{S}_{7,3} = \{\ -2\ \ 4\ \ -1\ \ 3\ \ -2\ \}$. Let $n = 4i+3 \geq 11$, first perform the move $\{\ -2\ \ 4i\ \}$. Then, ignore the peg at position $4i+3$ and sort the remaining $4i+2$ pegs using the solution $\mathcal{S}_{4i+2,3}$, see Section 3.3.2. Lemma 3.6 guarantees the validity of this solution $\mathcal{S}_{4i+3,3}$ which takes $\lceil \frac{4i+2}{2} \rceil + 1 = \lceil \frac{n}{2} \rceil$ moves. See Table 5 for the first solutions $\mathcal{S}_{n,3}$ for $n = 7, 11, 15$ and 19.

Table 5: First solutions for sorting $n$ pegs in $\lceil \frac{n}{2} \rceil$ Berge 3-moves for $n \equiv 3 \pmod 4$

$$
\begin{aligned}
\mathcal{S}_{7,3} &= \{\ \text{-2}\ \ 4\ \ \text{-1}\ \ 3\ \ \text{-2}\ \} \\
\mathcal{S}_{11,3} &= \{\ \text{-2}\ \ 8\ \ 1\ \ 7\ \ 0\ \ 5\ \ \text{-2}\ \} \\
\mathcal{S}_{15,3} &= \{\ \text{-2}\ \ 12\ \ 3\ \ 11\ \ 2\ \ 9\ \ 0\ \ 7\ \ \text{-2}\ \} \\
\mathcal{S}_{19,3} &= \{\ \text{-2}\ \ 16\ \ 5\ \ 15\ \ 4\ \ 13\ \ 2\ \ 11\ \ 0\ \ 9\ \ \text{-2}\ \}
\end{aligned}
$$

#### 3.3.4 Case $n \equiv 0 \pmod 4$

Although we found solutions in $\lceil \frac{n}{2} \rceil$ moves for $n \equiv 0 \pmod 4$, $20 \leq n \leq 48$, we could not find solutions in $\lceil \frac{n}{2} \rceil$ moves for all $n$. However, solutions $\bar{\mathcal{S}}_{4i,3}$ in $\lceil \frac{n}{2} \rceil + 1$ moves can be constructed as follows. Let $n = 4i \geq 16$, first perform the 2 moves $\{\ 4i+1\ \ 2\ \ 4i-3\ \}$. Then, ignore the six leftmost pegs, and the four rightmost pegs and sort the remaining $4i-10$ pegs using the solution $\mathcal{S}_{4i-10,3}$ shifted six spaces to the right, see Section 3.3.2. Finally, perform the 4 moves $\{\ 7\ \ 4i\ \ 6\ \ 2i+2\ \ 1\ \}$ to complete the solution $\bar{\mathcal{S}}_{4i,3}$. Lemma 3.6 guarantees the validity of this solution $\bar{\mathcal{S}}_{4i,3}$ which takes $2 + \lceil \frac{4i-10}{2} \rceil + 4 = \lceil \frac{n}{2} \rceil + 1$ moves. See Table 6 for the first solutions $\bar{\mathcal{S}}_{n,3}$ for $n = 16, 20$ and 24.

Table 6: First solutions for sorting $n$ pegs in $\lceil \frac{n}{2} \rceil + 1$ Berge 3-moves for $n \equiv 0 \pmod 4$

$$
\begin{aligned}
\bar{\mathcal{S}}_{16,3} &= \{\ 17\ \ 2\ \ 13\ \ 8\ \ 12\ \ 7\ \ 16\ \ 6\ \ 10\ \ 1\ \} \\
\bar{\mathcal{S}}_{20,3} &= \{\ 21\ \ 2\ \ 17\ \ 10\ \ 16\ \ 9\ \ 14\ \ 7\ \ 20\ \ 6\ \ 12\ \ 1\ \} \\
\bar{\mathcal{S}}_{24,3} &= \{\ 25\ \ 2\ \ 21\ \ 12\ \ 20\ \ 11\ \ 18\ \ 9\ \ 16\ \ 7\ \ 24\ \ 6\ \ 14\ \ 1\ \}
\end{aligned}
$$

While we could not exhibit solutions in $\lceil \frac{n}{2} \rceil$ moves for all $n \equiv 0 \pmod 4$, we believe that such solutions exist for $n \geq 20$, i.e., the proposed solutions $\bar{\mathcal{S}}_{4i,3}$ are not optimal, except for $\bar{\mathcal{S}}_{16,3}$. See Table 7 for optimal solutions in $\lceil \frac{n}{2} \rceil + 1$ moves for $n = 12$ and 16, and Table 8 for optimal solutions in $\lceil \frac{n}{2} \rceil$ moves for $n = 8, 20, 24, 28$ and 32.

Table 7: Solutions for sorting $n$ pegs in $\lceil \frac{n}{2} \rceil + 1$ Berge 3-moves for $n = 12$ and 16

$$\begin{array}{lcl} \mathcal{S}_{12,3} & = & \{\ 13\ \ 2\ \ 5\ \ 11\ \ 3\ \ 12\ \ 6\ \ 1\ \} \\ \bar{\mathcal{S}}_{16,3} & = & \{\ 17\ \ 2\ \ 13\ \ 8\ \ 12\ \ 7\ \ 16\ \ 6\ \ 10\ \ 1\ \} \end{array}$$

Table 8: Solutions for sorting $n$ pegs in $\lceil \frac{n}{2} \rceil$ Berge 3-moves for $n = 8, 20, 24, 28$ and 32

$$\begin{array}{lcl} \mathcal{S}_{8,3} & = & \{\ 9\ \ 2\ \ 7\ \ 3\ \ 9\ \} \\ \mathcal{S}_{20,3} & = & \{\ 21\ \ 2\ \ 7\ \ 12\ \ 17\ \} \cup \{\ 24\ \ 13\ \ 22\ \ 6\ \ 1\ \} \cup \{\ 17\ \ 8\ \ 24\ \} \\ \mathcal{S}_{24,3} & = & \{\ 25\ \ 6\ \ 13\ \ 18\ \} \cup \{\ \text{-}2\ \ 4\ \ 8\ \ 24\ \ 14\ \ 22\ \} \cup \{\ 18\ \ 3\ \ 12\ \ \text{-}1\ \ 25\ \} \\ \mathcal{S}_{28,3} & = & \{\ 29\ \ 2\ \ 7\ \ 16\ \ 23\ \ 12\ \} \cup \{\ 32\ \ 17\ \ 30\ \ 25\ \ 21\ \ 6\ \ 1\ \} \cup \{\ 12\ \ 23\ \ 8\ \ 32\ \} \\ \mathcal{S}_{32,3} & = & \{\ 33\ \ 2\ \ 7\ \ 12\ \ 17\ \ 24\ \} \cup \{\ 36\ \ 6\ \ 31\ \ 13\ \ 29\ \ 19\ \ 1\ \} \cup \{\ 24\ \ 11\ \ 35\ \ 18\ \ 28\ \ 4\ \} \end{array}$$

# 4 Related Questions

Other extensions of Berge's original questions include sorting any $n$ string:

($a_1$) Besides the alternating string, which other string requires exactly $h(n, k)$ Berge $k$-moves?

($a_2$) What is the minimum number of Berge $k$-moves required to sort any $n$ string?

($a_3$) Given a pair of strings, can we rearrange one into the other by Berge $k$-moves?

Associating the white and black colors to 0 and 1, the original $\{0, 1\}$-valued string could be generalized to $\{0, 1, \ldots, m\}$-valued strings where $m$ is the number of colors; the final string being $0 \ldots 0\, 1 \ldots 1 \ldots m \ldots m$:

($b_1$) What is the minimum number of Berge $k$-moves required to sort a string consisting of $m$ different integers - each integer being represented by the same number of pegs?

($b_2$) In particular, what is the minimum number of Berge $k$-moves required to sort a string consisting of $n$ different integers.

Generalizing to moves of $k$-by-$k$ blocks in the plane could also be considered. Similar questions were raised for 2-moves in [1].

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Table 9: Values of $h(n,k) - \lceil \frac{n}{2} \rceil$ for $k \leq 14$ and $n \leq 50$

| $n \backslash k$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 5 | 0 | 0 | – | – | – | – | – | – | – | – | – | – | – |
| 6 | 0 | 0 | 3 | – | – | – | – | – | – | – | – | – | – |
| 7 | 0 | 0 | 0 | 2 | – | – | – | – | – | – | – | – | – |
| 8 | 0 | 0 | 1 | 2 | 3 | – | – | – | – | – | – | – | – |
| 9 | 0 | 0 | 0 | 1 | 2 | 3 | – | – | – | – | – | – | – |
| 10 | 0 | 0 | 1 | 1 | 1 | 3 | 6 | – | – | – | – | – | – |
| 11 | 0 | 0 | 0 | 1 | 1 | 2 | 4 | 6 | – | – | – | – | – |
| 12 | 0 | 1 | 1 | 1 | 1 | 2 | 3 | 5 | 10 | – | – | – | – |
| 13 | 0 | 0 | 0 | 1 | 1 | 1 | 2 | 3 | 4 | 11 | – | – | – |
| 14 | 0 | 0 | 0 | 1 | 2 | 2 | 2 | 2 | 4 | 6 | 15 | – | – |
| 15 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 2 | 2 | 4 | 7 | 14 | – |
| 16 | 0 | 1 | 0 | 1 | 1 | 0 | 2 | 2 | 3 | 3 | 5 | 9 | 21 |
| 17 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 2 | 2 | 3 | 5 | 9 |
| 18 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 2 | 3 | 3 | 4 | 7 |
| 19 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 2 | 3 | 4 |
| 20 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 3 | 3 | 4 |
| 21 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 2 | 2 | 3 |
| 22 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 2 | 1 | 2 | 3 |
| 23 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 2 |
| 24 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 2 |
| 25 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 2 |
| 26 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 1 | 1 | 2 |
| 27 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 |
| 28 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 |
| 29 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 |
| 30 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 |
| 31 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 32 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 33 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 34 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 35 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 36 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 37 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 38 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 39 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 40 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 41 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 42 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 43 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 44 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 45 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 46 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 47 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 48 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 49 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 50 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

## References

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Antoine Deza, William Hua
Advanced Optimization Laboratory, Department of Computing and Software,
McMaster University, Hamilton, Ontario, Canada.
*Email*: deza@mcmaster.ca.