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Authors:

Antoine Deza and Gabriel Indik

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Antoine Deza Gabriel Indik

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Abstract

The metric polytope met_n is the polyhedron associated with all semimetrics on n nodes and defined by the triangle inequalities $x_{ij} - x_{ik} - x_{jk} \leq 0$ and $x_{ij} + x_{ik} + x_{jk} \leq 2$ for all triples i, j, k of $\{1, \ldots, n\}$. In 1992 Monique Laurent and Svatopluk Poljak conjectured that every fractional vertex of the metric polytope is adjacent to some integral vertex. The conjecture holds for $n \leq 8$ and, in particular, for the 1 550 825 600 vertices of met_8. While the overwhelming majority of the known vertices of met_9 satisfy the conjecture, we exhibit a fractional vertex not adjacent to any integral vertex.

1 Introduction and Notation

The $\binom{n}{2}$-dimensional *cut cone* Cut_n is usually introduced as the conic hull of the incidence vectors of all the cuts of the complete graph on n nodes. More precisely, given a subset S of $V_n = \{1, \ldots, n\}$, the *cut* determined by S consists of the pairs (i, j) of elements of V_n such that exactly one of i, j is in S. By $\delta(S)$ we denote both the cut and its incidence vector in $\mathbb{R}^{\binom{n}{2}}$, i.e., $\delta(S)_{ij} = 1$ if exactly one of i, j is in S and 0 otherwise for $1 \leq i < j \leq n$. We use the term cut for both the cut itself and its incidence vector, so $\delta(S)_{ij}$ are coordinates of a point in $\mathbb{R}^{\binom{n}{2}}$.

The cut cone Cut_n is the conic hull of all $2^{n-1} - 1$ nonzero cuts, and the *cut polytope* cut_n is the convex hull of all 2^{n-1} cuts. The cut cone and a relaxation, the *metric cone* Met_n, can also be defined in terms of finite metric spaces in the following way. For all triples $\{i, j, k\} \subset V_n$, we consider the following inequalities.

$$\begin{aligned} x_{ij} - x_{ik} - x_{jk} &\leq 0, & (1)\\ x_{ij} + x_{ik} + x_{jk} &\leq 2. & (2) \end{aligned}$$

(1) specify the $3\binom{n}{3}$ facets of the cone Met_n of *semimetrics* on V_n; that is, of functions $x : V_n \times V_n \to \mathbb{R}_+$ satisfying $x_{ij} = x_{ji}$, $x_{ii} = 0$, and the *triangle* inequalities (1). While x is a *metric* only when $x_{ij} > 0$ for all $i \neq j$, we will follow the usual convention and call Met_n the *metric cone*.

It is well-known that Cut_n is the conic hull of all, up to a constant multiple, $\{0, 1\}$-valued extreme rays of Met_n. The cuts satisfy the *perimeter* inequalities (2) which can also be obtained from (1) by the *switching* operation, see Section 4. Bounding Met_n by the $\binom{n}{3}$ facets induced by (2), we obtain a natural relaxation of cut_n, the *metric polytope* met_n, so that cut_n is the convex hull of all $\{0,1\}$-valued vertices of met_n.

One of the motivations for the study of these polyhedra comes from their applications in combinatorial optimization, the most important being the MAXCUT and multicommodity flow problems. We refer to Deza and Laurent [9] and to Poljak and Tuza [15] for a detailed study of those polyhedra and their applications in combinatorial optimization.

2 A Counterexample to the Dominating Set Conjecture

Laurent and Poljak [14] conjectured that every fractional vertex of the metric polytope met_n is adjacent to some integral vertex, i.e., to a cut. Since we have $\mathrm{met}_3 = \mathrm{cut}_3$ and $\mathrm{met}_4 = \mathrm{cut}_4$, the conjecture, which is commonly called the dominating set conjecture, is obviously true for the 4 vertices of met_3 and for the 8 vertices of met_4. The conjecture holds for the 32 vertices of met_5 and the 544 vertices of met_6 as well as for several classes of vertices of met_n, see [12]. The conjecture was further substantiated by the computation of met_7 and met_8. The 275 840 vertices of met_7 and the 1 550 825 600 vertices of met_8 are adjacent to a cut, see [4, 5, 6].

While the overwhelming majority of the known vertices of met_9 satisfy the dominating set conjecture, few counterexamples were found. In particular, we exhibit the following fractional vertex not adjacent to any integral vertex.

Proposition 2.1. *The neighbors of the fractional vertex* $\frac{1}{9}(2, 2, 3, 3, 4, 4, 5, 5, 4, 3, 5, 6, 6, 3, 3, 5, 5, 2, 4, 3, 5, 6, 3, 3, 6, 6, 5, 3, 2, 6, 6, 3, 3, 5, 3, 4)$ *of the metric polytope* met_9 *are all fractional.*

The vertex given in Proposition 2.1, as well as other vertices not adjacent to any cut, were found by an extensive computer search of the vertices of the 36-dimensional metric polytope met_9, see [3]. Note that while finding a vertex providing a counterexample to the dominating set conjecture is computationally challenging, to verify that a given vertex is indeed not adjacent to a cut is easy if the vertex is quasi-simple, i.e., if the incidence of the given vertex is equal to the dimension plus one. For example, one can easily check, see Section 4, that the vertex given in Proposition 2.1 satisfies with equalities 37 of the 336 inequalities defining met_9 and is adjacent to 37 vertices which are all fractional.

3 Related Questions

3.1 The diameter of the metric polytope

Since any pair of cuts forms an edge of met_n, the dominating set conjecture would imply that the diameter $\delta(\mathrm{met}_n)$ of the metric polytope satisfies $\delta(\mathrm{met}_n) \leq 3$. We recall that the diameter of a polytope P is the smallest number $\delta(P)$ such that any two vertices of P can be connected by a path with at most $\delta(P)$ edges. We have $\delta(\mathrm{met}_3) = \delta(\mathrm{met}_4) = 1$, $\delta(\mathrm{met}_5) = \delta(\mathrm{met}_6) = 2$ and $\delta(\mathrm{met}_7) = \delta(\mathrm{met}_8) = 3$. While the diameter of the restriction of met_9 to its known vertices appears to be less than 3, it is not clear that the diameter of met_n is bounded by a constant.

3.2 The no-cut set conjecture

Conjecture 3.1. [6] *For $n \geq 6$, the restriction of the metric polytope* met_n *to its fractional vertices is connected.*

Conjecture 3.1 can be seen as complementary to the dominating set conjecture both graphically and computationally: For any pair of vertices, while the dominating set conjecture implies that there is a path made of cuts joining them, i.e., the cut vertices form a dominating set, Conjecture 3.1 means that there is a path made of non-cut vertices joining them, i.e., the cut vertices do not form a cut-set. On the other hand, while the dominating set conjecture means that the enumeration of the extreme rays of the metric cone Met_n is enough to obtain the vertices of the metric polytope met_n, Conjecture 3.1 means that we can obtain the vertices of met_n without enumerating the extreme rays of Met_n.

4 Counterexamples Generation and Verification

One important feature of the metric and cut polyhedra is their very large symmetry group. We recall that the symmetry group $Is(P)$ of a polyhedron P is the group of isometries preserving P and that an isometry is a linear transformation preserving the Euclidean distance. For $n \geq 5$, the symmetry groups of the polytopes met_n and cut_n are isomorphic and induced by permutations on V_n and *switching reflections by a cut*, see [8], and the symmetry groups of the cones Met_n and Cut_n are isomorphic to $Sym(n)$, see [7]. Given a cut $\delta(S)$, the switching reflection $r_{\delta(S)}$ is defined by $y = r_{\delta(S)}(x)$ where $y_{ij} = 1 - x_{ij}$ if $(i,j) \in \delta(S)$ and $y_{ij} = x_{ij}$ otherwise.

subsectionCounterexamples generation The vertices of met_n are partitioned into orbits under the action of the symmetry group $Is(\mathrm{met}_n)$. Using a parallel implementation of an orbitwise enumeration algorithm, 1 056 368 orbits of vertices of met_9 were computed on Shared Hierarchical Academic Research Computing Network (SHARCNET) clusters. Among these 1 056 368 orbits, 1 221 provide counterexamples to the dominating set conjecture, including 483 made of quasi-simplicial vertices. As the dimension of met_9 is 36, these quasi-simplicial counterexamples satisfy with equality exactly 37 inequalities. The remaining 738 orbits providing counterexamples to the dominating set conjecture are made of vertices satisfying with equality at least 38 inequalities. Some of these counterexamples have a relatively large incidence and adjacency: For example, the vertex $\frac{1}{9}$(1, 2, 3, 3, 4, 4, 4, 6, 3, 4, 4, 3 3, 5, 7, 5, 5, 6, 6, 2, 4, 6, 3, 3, 3, 3, 3, 3, 3, 3, 6, 6, 4, 6, 4, 2) satisfies with equality 44 inequalities and has 84 fractional adjacent vertices, and the vertex $\frac{1}{12}$(3, 3, 3, 6, 7, 7, 7, 7, 6, 6, 3, 4, 4, 4, 4, 6, 3, 4, 4, 4, 4, 9, 4, 4, 8, 8, 7, 7, 7, 7, 8, 4, 4, 4, 4, 8) satisfies with equality 43 inequalities and has 202 fractional adjacent vertices. See [3] for a complete list of the known counterexamples.

4.1 Counterexamples verification

For a given quasi-simple vertex, one can easily verify that all the adjacent vertices are fractional by performing 3 simple computations which we illustrate using the vertex given in Proposition 2.1.

(*i*) Check which of the 336 inequalities of met_9 are satisfied with equality by the given vertex. For the vertex given in Proposition 2.1, we obtain the 37 inequalities given in Section 4.2.1.

(*ii*) Compute the pointed cone formed by the active inequalities identified in (*i*). For the vertex given in Proposition 2.1, we obtain a quasi-simplicial cone with 37 extreme rays.

(*iii*) From the given vertex, follow each extreme ray of the cone computed in (*ii*) until at least one of the inequalities defining met_9 is violated. The obtained points are the adjacent vertices. For the vertex given in Proposition 2.1, we obtain the 37 adjacent vertices given in Section 4.2.2. As these 37 adjacent vertices are all fractional, the vertex given in Proposition 2.1 is indeed a counterexample.

Note that, while the computation (*ii*) can be extremely expensive for a highly degenerate vertex in high dimension, it can be done efficiently if the given vertex is quasi-simple. It takes less than a second of CPU time for the vertex given in Proposition 2.1 using enumeration packages such as *lrs* [2] or *cdd* [11]. Computations (*i*) and (*iii*) are straightforward and take less than a second of CPU time.

4.2 Given counterexample incidence and adjacency lists

4.2.1 Given counterexample incidence list

The vertex given in Proposition 2.1 satisfies with equalities the following 37 inequalities of met_9: $\Delta_{6,7,\bar{9}}$, $\Delta_{5,\bar{8},9}$, $\Delta_{5,\bar{7},9}$, $\Delta_{\bar{5},7,8}$, $\Delta_{5,6,\bar{8}}$, $\Delta_{4,\bar{7},9}$, $\Delta_{4,\bar{6},9}$, $\Delta_{4,\bar{6},8}$, $\Delta_{\bar{4},6,7}$, $\Delta_{4,5,9}$, $\Delta_{4,5,\bar{7}}$, $\Delta_{3,\bar{6},9}$, $\Delta_{\bar{3},6,7}$, $\Delta_{3,5,\bar{8}}$, $\Delta_{3,4,\bar{6}}$, $\Delta_{2,7,\bar{9}}$, $\Delta_{2,6,\bar{9}}$, $\Delta_{2,6,\bar{8}}$, $\Delta_{2,6,7}$, $\Delta_{2,5,\bar{8}}$, $\Delta_{\bar{2},4,9}$, $\Delta_{\bar{2},4,8}$, $\Delta_{2,\bar{4},7}$, $\Delta_{2,\bar{4},6}$, $\Delta_{2,\bar{3},6}$, $\Delta_{1,\bar{5},8}$, $\Delta_{\bar{1},4,5}$, $\Delta_{1,\bar{3},8}$, $\Delta_{1,\bar{3},6}$, $\Delta_{\bar{1},3,5}$, $\Delta_{\bar{1},3,4}$, $\Delta_{1,\bar{2},9}$, $\Delta_{1,\bar{2},8}$, $\Delta_{\bar{1},2,7}$, $\Delta_{\bar{1},2,6}$, $\Delta_{\bar{1},2,5}$, $\Delta_{\bar{1},2,3}$ where the triangle inequality (1) and the perimeter inequality (2) are respectively denoted by $\Delta_{i,j,\bar{k}}$ and $\Delta_{i,j,k}$.

4.2.2 Given counterexample adjacency list

The vertex given in Proposition 2.1 is adjacent to the following 37 fractional vertices of met_9:

$\frac{1}{3}(0,1,1,1,2,2,1,1,1,1,1,2,2,1,1,2,0,1,1,0,2,2,1,1,2,2,1,1,0,2,2,1,1,1,1,2)$
$\frac{1}{3}(0,1,1,1,2,2,1,1,1,1,1,2,2,1,1,2,2,1,1,2,2,2,1,1,2,2,1,1,0,2,2,1,1,1,1,2)$
$\frac{1}{3}(1,0,2,0,1,1,0,2,1,1,1,2,2,1,1,2,0,1,1,0,2,2,1,1,2,2,1,1,0,2,2,1,1,1,1,2)$
$\frac{1}{3}(1,1,0,1,1,1,2,2,2,1,2,2,2,1,1,1,2,0,2,1,1,1,1,1,2,2,2,0,1,1,2,1,1,1,1,0)$
$\frac{1}{3}(1,1,0,1,1,1,2,2,2,1,2,2,2,1,1,1,2,0,2,1,1,1,1,1,2,2,2,2,1,3,2,1,1,3,1,2)$

$\frac{1}{3}(1,1,1,1,1,1,2,2,1,1,2,2,2,1,1,2,2,1,1,1,2,2,1,1,2,2,2,1,1,2,2,1,1,2,1,1)$
$\frac{1}{3}(1,1,1,1,1,1,2,2,2,1,2,2,2,1,1,1,2,0,2,1,1,2,1,1,2,2,2,1,1,2,2,1,1,2,1,1)$
$\frac{1}{3}(1,1,1,1,1,1,2,2,2,1,2,2,2,1,1,2,2,1,1,1,2,2,1,1,2,2,2,1,1,2,2,1,1,2,1,1)$
$\frac{1}{3}(1,1,1,1,2,1,2,2,1,1,2,2,2,1,1,2,2,1,1,1,2,2,1,1,2,2,2,1,1,2,2,1,1,2,1,1)$
$\frac{1}{3}(1,1,1,1,2,1,2,2,2,1,2,2,2,1,1,2,2,1,1,1,2,2,1,1,2,2,2,1,1,2,2,1,1,2,1,1)$
$\frac{1}{4}(1,1,1,1,1,2,2,2,2,2,2,2,3,1,1,2,2,0,3,1,1,2,2,1,3,3,2,1,1,3,3,1,1,2,2,2)$
$\frac{1}{4}(1,1,1,1,2,2,2,2,2,2,2,3,3,1,1,2,2,1,1,1,3,2,1,1,3,3,3,1,1,3,2,2,2,2,2,2)$
$\frac{1}{6}(1,1,2,2,3,3,3,3,2,2,3,4,4,2,2,3,3,2,2,2,4,4,2,2,4,4,3,2,1,4,4,2,2,3,2,3)$
$\frac{1}{6}(1,1,3,1,3,3,2,3,2,2,2,4,4,1,2,4,2,2,2,1,4,4,2,2,3,4,4,2,1,4,4,3,2,3,2,3)$
$\frac{1}{6}(1,1,3,2,3,3,3,3,2,2,3,4,4,2,2,4,3,2,2,2,4,4,2,2,4,4,3,2,1,4,4,2,2,3,2,3)$
$\frac{1}{6}(1,1,3,2,3,3,3,3,2,2,3,4,4,2,2,4,3,2,2,2,4,5,2,2,4,4,3,3,1,3,4,2,2,4,2,2)$
$\frac{1}{6}(2,1,2,2,2,2,4,4,3,2,4,4,4,2,2,3,3,1,3,3,3,4,2,2,4,4,4,2,2,4,4,2,2,4,2,2)$
$\frac{1}{7}(1,1,3,2,3,3,3,3,2,2,3,4,4,2,2,4,3,2,2,2,4,5,2,2,4,4,3,3,1,5,4,2,2,4,2,4)$
$\frac{1}{7}(1,1,3,2,3,4,3,3,2,2,3,4,5,2,2,4,3,2,3,2,4,5,2,3,4,4,3,2,1,5,5,2,2,3,3,4)$
$\frac{1}{7}(1,1,3,2,3,4,3,4,2,2,3,4,5,2,3,4,3,2,3,2,5,5,2,3,4,5,3,2,1,4,5,2,3,3,2,3)$
$\frac{1}{7}(2,2,2,3,3,2,5,4,4,2,5,5,4,3,2,4,5,1,4,3,4,5,3,2,5,4,4,3,2,5,5,2,3,5,2,3)$
$\frac{1}{7}(2,2,2,3,3,3,5,5,4,2,5,5,5,3,3,4,5,1,3,3,3,5,3,3,5,5,4,2,2,4,4,2,2,4,2,2)$
$\frac{1}{9}(1,2,4,2,5,5,4,4,3,3,3,6,6,3,3,6,4,3,3,2,6,6,3,3,6,6,5,3,2,6,6,3,3,5,3,4)$
$\frac{1}{9}(2,2,3,3,4,4,5,3,4,3,5,6,6,3,3,5,5,2,4,3,5,6,3,3,6,6,5,3,2,6,6,3,3,5,3,4)$
$\frac{1}{9}(2,2,3,3,4,4,5,5,4,3,5,6,6,3,3,5,5,2,4,3,3,6,3,3,6,6,5,3,2,6,6,3,3,5,3,4)$
$\frac{1}{9}(2,2,3,3,4,4,5,5,4,3,5,6,6,3,3,5,5,2,4,3,5,6,3,3,6,6,5,3,2,6,6,3,3,5,3,6)$
$\frac{1}{9}(2,2,3,3,4,4,5,5,4,3,5,6,6,3,3,5,5,2,4,3,5,6,3,3,6,6,3,3,2,6,6,3,3,5,3,4)$
$\frac{1}{9}(2,2,3,3,4,4,5,5,4,3,5,6,6,3,3,5,5,2,4,3,5,6,3,3,6,6,5,3,2,6,6,3,3,3,3,4)$
$\frac{1}{9}(2,2,3,3,4,4,5,5,4,3,5,6,6,3,3,5,5,2,6,3,5,6,3,3,6,6,5,3,2,6,6,3,3,5,3,4)$
$\frac{1}{9}(2,2,3,3,4,6,5,5,4,3,5,6,6,3,3,5,5,2,4,3,5,6,3,3,6,6,5,3,2,6,6,3,3,5,3,4)$
$\frac{1}{9}(3,2,2,4,3,3,6,6,5,3,5,6,6,3,3,4,6,1,5,4,4,6,3,3,6,6,5,3,2,6,6,3,3,5,3,4)$
$\frac{1}{10}(2,2,4,3,5,4,5,5,4,4,5,7,6,3,3,6,5,3,4,3,7,7,3,4,7,7,6,3,2,6,7,4,4,5,3,4)$
$\frac{1}{10}(2,2,4,3,5,5,5,6,4,4,5,7,7,3,4,6,5,3,3,3,6,7,3,3,7,6,6,4,2,7,6,4,3,6,3,5)$
$\frac{1}{10}(3,3,2,4,4,3,7,7,6,3,7,7,6,4,4,5,7,1,6,4,4,6,4,3,7,7,6,3,3,7,7,3,3,6,4,4)$
$\frac{1}{12}(3,3,3,5,5,5,7,7,6,4,8,8,8,4,4,6,8,2,6,4,6,8,4,4,8,8,8,4,4,8,8,4,4,8,4,4)$
$\frac{1}{12}(3,3,3,5,5,5,8,7,6,4,8,8,8,5,4,6,8,2,6,5,6,8,4,4,7,8,6,4,3,8,8,3,4,7,4,5)$
$\frac{1}{12}(3,3,3,5,5,5,8,7,6,4,8,8,8,5,4,6,8,2,6,5,6,8,4,4,9,8,8,4,3,8,8,5,4,7,4,5)$

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