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Abstract

One can easily characterize the closure of the hyperbolicity cone K_p associated to an arbitrary hyperbolic polynomial p in terms of finitely many polynomial inequalities and construct a logarithmic self-concordant barrier functional for K_p. In contrast, little is known about its dual cone ${K_p}^*$.

Elementary symmetric polynomials can be thought of as derivative polynomials of $E_n(x) = \prod_{i=1\ldots n} x_i$. Their associated hyperbolicity cones give a natural sequence of relaxations for $\mathbb{R}^n_+ = K_{E_n}$. Once the recursive structure of these cones is established, we give an algebraic characterization for the dual cone associated with $E_{n-1}(x) = \sum_{1\leq i\leq n} \prod_{j\neq i} x_j$ and show how one can easily construct a self-concordant barrier functional for this cone.

1 Introduction

Let X ($\equiv \mathbb{R}^n$) be a finite dimensional real vector space equipped with an inner product $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{R}$. Denote $\mathbf{1} \in \mathbb{R}^n$ – vector of all ones.

Hyperbolic polynomials and the associated hyperbolicity cones have origins in partial differential equations (see [12]). Recently, these structures have drawn considerable attention in the optimization community as well (see [6], [2], [14], [7]). It turns out that most of interior point methods (IPM) theory (see [11], [13]) applies naturally to the class of conic programming problems (CP)[1] arising from hyperbolicity cones (what we refer to as *hyperbolic programming program*). In particular, linear programming, second-order conic programming and positive semi-definite programming are themselves instances of conic programming problems of this kind.

Definition 1.1. For a cone $K \subseteq \mathbb{R}^n$, the *dual cone* is defined as $K^* = \{y \in \mathbb{R}^n : \forall x \in K, \langle x, y\rangle \geq 0\}$

[1]A *conic program* is an optimization problem of the form $\{\inf_x \langle c, x\rangle : Ax = b, x \in K\}$ with $K \subset \mathbb{R}^n$ being a closed convex cone, $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ and $A \in \mathbb{R}^{m\times n}$. It is well known that any convex optimization problem can be recast as conic programming problem.

Often, the dual cone provides much information about the original CP (indeed, the most successful IPM algorithms are the so-called primal-dual algorithms, which follow the so-called central paths in K and K^* simultaneously). Hence, the understanding of the structure of both the primal cone and the dual cone for a given conic programming problem usually plays a very important role in achieving greater computational efficiency in solving these optimization problems.

While a simple characterization for the hyperbolicity cones as a set of polynomial inequalities is known, little is known regarding the algebraic structure of their dual cones, with some exceptions (see [5]). That the dual cones can be represented by systems of polynomial inequalities follows from Tarski's establishment of quantifier elimination methods (see [4]). These methods, however, give little insight into the precise algebraic structure of the dual cones, because the methods result in extremely complicated systems of polynomial inequalities, even for hyperbolic polynomials in 3 variables.

We attempt to understand the structure of hyperbolicity cones associated with elementary symmetric polynomials (which is an important family of hyperbolicity cones) and the structure of the associated dual cones.

2 Hyperbolic programming

Definition 2.1. Let $p : X \to \mathbb{R}$ be a homogeneous polynomial of degree $m \in \mathbb{N}$ (i.e., $p(tx) = t^m p(x)$, $\forall t \in \mathbb{R}$ and every $x \in X$) and $d \in X$ is such that $p(d) \neq 0$. Then p is *hyperbolic with respect to* d if the univariate polynomial $\lambda \mapsto p(x - \lambda d)$ has all roots real for every $x \in X$.

Examples:

- $X = \mathbb{R}^n$, $d = \mathbf{1}$. The n^{th} elementary symmetric polynomial, $E_n(x) = \prod_{i=1}^n x_i$, is a hyperbolic polynomial with respect to d (for $E_n(x - \lambda\mathbf{1}) = \prod_{i=1}^n (x_i - \lambda)$),
- $X = \mathbb{S}^k$, the space of real symmetric $k \times k$ matrices, $d = I$ (identity matrix). The determinant, $\det(x)$, is a hyperbolic polynomial in direction d (for the eigenvalues of $x \in \mathbb{S}^k$ are the roots of $\det(x - \lambda I)$ and are real).

The roots are called the *eigenvalues* of x (in direction d), terminology motivated by the last example. We denote the eigenvalues by

$$\lambda_1(x) \leq \lambda_2(x) \leq \cdots \lambda_m(x)$$

or simply $\lambda(x)(\in \mathbb{R}^m)$.

We introduce sums of the smallest k eigenvalues as follows

$$s_k := \sum_{i=1}^{k} \lambda_i$$

(denoting $s(x) = (s_1, s_2, \ldots, s_m) \in \mathbb{R}^m$).

Fact 2.2 ([2]; also see [12], [14] for $s_1(x) \equiv \lambda_1(x)$). *$s_k(x)$ is a concave function $\forall k$.*

Definition 2.3. The *hyperbolicity cone of* p with respect to d, written $C(p,d)$, is the set $\{x \in X : p(x+td) \neq 0, \forall t \geq 0\}$.

Note that $C(p,d) = \{x \in X : \lambda_1(x) > 0\}$.
Examples:

- $X = \mathbb{R}^n$, $d = \mathbf{1}$, $p(x) = E_n(x)$, then $C(p,d) = \mathbb{R}^n_{++}$ (the positive orthant),
- $X = \mathbb{S}^k$, $d = I$, $p(x) = \det(x)$, then $C(p,d) = \mathbb{S}^k_{++}$ (the cone of positive definite matrices).

Fact 2.4 ([12]). *Given a pair p, d:*

(i) $d \in C(d)$,

(ii) $C(d)$ *is an open convex cone,*

(iii) $clC(d) = \{x \in X : \lambda_1(x) \geq 0\}$,

(iv) if $c \in C(d)$*, then* p *is hyperbolic in direction* c *and* $C(c) = C(d)$.

3 Derivative polynomials and primal cone characterization

3.1 Derivative polynomials

Given a hyperbolic polynomial p (of degree m) in direction d, denote

$$p'(d,x) = \frac{\partial}{\partial t} p(x+td)|_{t=0} = \nabla_x p(x)^T d$$

We will refer to p' as the "derivative polynomial of p (with respect to d)" and usually will write $p'(x)$ instead of $p'(d,x)$ omitting (the parameter) d for simplicity of notation (when the choice of d is obvious). By the root interlacing property for the polynomials with all real roots (by continuity between any two roots of $t \mapsto p(x+td)$ there is a root of $\frac{\partial}{\partial t} p(x+td)$) it follows that $p'(x)$ is also hyperbolic in direction d.

Similarly, (for a fixed hyperbolicity direction d) we can define higher derivatives $p'', p''', \ldots, p^{(m)}$. Note that since p was assumed to be of degree m, $p^{(m-1)}$ is linear and $p^m(x)$ is constant.

Example: $X = \mathbb{R}^n$, $d \in \mathbb{R}^n_{++}$, $p(x) = E_n(x)$. Then by easy computation one can show that

$$E_n^{(k)}(x) = (k!) E_n(d) E_{n-k}\left(\left[\frac{x_1}{d_1}, \frac{x_2}{d_2}, \ldots, \frac{x_n}{d_n}\right]\right)$$

where $E_k(x)$ is the k^{th} elementary symmetric polynomial[2].

[2] $E_k(x) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \prod_{j=1}^k x_{i_j}$

Remark 3.1. It should be noted that the elementary symmetric polynomials in the example above also play an important role in representing the derivative polynomials via the eigenvalues at a point $x \in X$. As a consequence of homogeneity (see, for example, [14]) it follows that

$$p^{(k)}(x) = (k!)p(d)E_{m-k}(\lambda)$$

3.2 Cone characterization

Denote $K_{p,d} := \text{cl}C(p, d)$, the closure of hyperbolicity cone. When the choice of d is obvious, we will omit it from the notation, thus writing just K_p.

We present a well-known result giving one particular characterization of $K_{p,d}$.

Theorem 3.2 (see, for example, [14]). *Suppose p is a hyperbolic polynomial of degree m with respect to d, (w.l.o.g.) $p(d) > 0$, and $p', p'', \ldots$ are defined as above. Then*

$$K_{p,d} = \{x \in \mathbb{R}^n : p(x) \geq 0, p'(x) \geq 0, p''(x) \geq 0, \ldots, p^{(m-1)}(x) \geq 0\}$$

Corollary 3.3. *Given a pair p, d we have the following cone inclusions:*

$$K_{p,d} \subseteq K_{p',d} \subseteq \cdots \subseteq K_{p^{(m-1)},d}$$

In particular, for $X = \mathbb{R}^n$, $d \in \mathbb{R}^n_{++}$,

$$\mathbb{R}^n_+ = K_{E_n,d} \subseteq K_{E_n^{(1)},d} \subseteq \cdots \subseteq K_{E_n^{(n-1)},d}$$

giving us a natural sequence of relaxations of the nonnegative orthant.

Corollary 3.4. *Given a pair p, d, $p(d) > 0$, the boundary of $K_{p,d}$ satisfies*

$$\partial K_{p,d} = \{x \in \mathbb{R}^n : p(x) = 0, p'(x) \geq 0, \ldots, p^{(m-1)}(x) \geq 0\}$$

Proof. Follows from the root interlacing property for hyperbolic polynomials. □

Proposition 3.5. *Assume $1 \leq r \leq (m-2)$. If $x \in K_{p^{(r)},d}$ and $p^{(r+1)}(x) = 0$ (that is $x \in K_{p^{(r+1)},d}$), then $x \in K_{p,d}$.*

Proof. By the root interlacing property for polynomials with all real roots it follows that $t = 0$ is a multiple root of $t \mapsto p^{(r)}(x + td)$ of multiplicity $l \geq 2$. Therefore, 0 is a root of multiplicity $(l+1)$ for $t \mapsto p^{(r-1)}(x + td)$, and so on, until we get to p itself. Since 0 is the right-most root for $t \mapsto p^{(r)}(x + td)$ ($x \in K_{p^{(r)},d}$), it is also the right-most root $t \mapsto p(x + td)$ (by counting roots). So $x \in K_{p,d}$ (in fact, $x \in \partial K_{p,d}$). □

4 Semi-definite representability and the dual cones

It has been long hypothesized that the hyperbolicity cones and the cone of positive semi-definite matrices have strong connections. In particular one of the open questions is whether the hyperbolicity cones are more general than the linear sections of $\mathbb{S}^d_+$ (and consequently, whether hyperbolic programming is any more general than SDP).

In 1958, Peter Lax conjectured that each hyperbolic polynomial $p(x)$ in 3 variables satisfies $p(x) = \det(x_1 A + x_2 B + x_3 C)$, for some $A, B, C \in \mathbb{S}^d$, consequently each hyperbolicity cone in 3 variables can be realized as the intersection of $\mathbb{S}^d_+$ with an affine subspace of $\mathbb{S}^d$. The conjecture was recently established affirmatively in [10] – as a corollary to work of[8]. It remains open whether similar representations hold for hyperbolicity cones in more than three variables, although such representations have been established for important broad families of hyperbolicity cones (in particular, the so-called homogeneous cones,[5]).

It turns out that this representation also explains the structure of the corresponding dual cones (under some mild assumptions).

Definition 4.1 (as in [3])**.** The (convex) set $Y \subseteq \mathbb{R}^n$ is said to be *SDR (positive semi-definite representable)* if

$$x \in Y \Leftrightarrow \mathcal{A}\begin{bmatrix} x \\ u \end{bmatrix} + B \succeq 0 \quad \text{(is positive semi-definite), for some} \quad u \in \mathbb{R}^m$$

where $B \in \mathbb{S}^k$ and $\mathcal{A} : \mathbb{R}^{n+m} \to \mathbb{S}^k$ can be written as

$$\mathcal{A}\begin{bmatrix} x \\ u \end{bmatrix} = \sum_{i=1}^{n} x_i A_i + \sum_{j=1}^{m} u_j B_j$$

with $A_i, B_j \in \mathbb{S}^k$.

Fact 4.2. *If X is SDR then so is an affine image of X.*

Proof. Can easily show by switching to the appropriate basis in $\mathbb{S}^k$, see [3]. □

We give a SDR analogue of a Second-Order Cone Representability Theorem in [3].

Proposition 4.3. *If $K \subset \mathbb{R}^n$ is a (closed convex) cone with nonempty interior and*

$$K = \left\{ x \in \mathbb{R}^n : \exists u \quad \textit{such that} \quad \mathcal{A}\begin{bmatrix} x \\ u \end{bmatrix} + B \succeq 0 \right\}$$

then its dual satisfies

$$K^* = \left\{ y \in \mathbb{R}^n : \exists \Lambda \quad \textit{such that} \quad \begin{pmatrix} y \\ 0 \end{pmatrix} = \mathcal{A}^* \Lambda, \langle B, \Lambda \rangle \leq 0, \Lambda \succeq 0 \right\}$$

where $\mathcal{A}^* : \mathbb{S}^k \to \mathbb{R}^{n+m}$ is the adjoint of $\mathcal{A}$, defined as

$$\mathcal{A}^* \Lambda = (\langle A_1, \Lambda \rangle, \langle A_2, \Lambda \rangle, \cdots, \langle A_n, \Lambda \rangle, \langle B_1, \Lambda \rangle, \cdots, \langle B_m, \Lambda \rangle)$$

Proof. Considering the primal-dual pair

$$\inf_{x,u}\left\{\begin{bmatrix} y \\ 0 \end{bmatrix}^T\begin{bmatrix} x \\ u \end{bmatrix} : \mathcal{A}\begin{bmatrix} x \\ u \end{bmatrix} + B \succeq 0\right\}$$

and

$$\sup_{\Lambda \succeq 0}\{\langle -B, \Lambda\rangle = -Tr(B\Lambda) : \langle A_i, \Lambda\rangle = y_i, i = 1\ldots n, \langle B_j, \Lambda\rangle = 0, j = 1\ldots m\}$$

by the Conic Duality Theorem ([13]) we conclude that $y \in K^*$, iff the first problem is bounded below by 0, and hence iff the second has a feasible solution with the value of at least 0. Thus

$$K^* = \{y : \exists \Lambda \succeq 0 \text{ s.t. } \langle A_i, \Lambda\rangle = y_i, i = 1\ldots n, \langle B_j, \Lambda\rangle = 0, j = 1\ldots m, \langle B, \Lambda\rangle \leq 0\}$$

□

5 Elementary symmetric polynomials and the ratio functional

For $0 \leq k \leq (n-1)$, $p(x) = E_n^{(k)}(x)$, $d \in \mathbb{R}^n_{++}$, $p'(x) = E_n^{(k+1)}(x)$ we extend the domain of concavity for the ratio functional $p(x)/p'(x)$ to $K_{p',d}$ (that this function is concave over $K_{p,d}$ follows from Theorem 3.8 in [2]).

Proposition 5.1 (J. Renegar). *Let $x \in \mathbb{R}^n, d = \mathbf{1} \in \mathbb{R}^n$, $p(x) = \prod_{i=1}^n x_i$, and $p'(x) = \sum_{i=1}^n \prod_{j\neq i} x_i$. Then*

$$q_n(x) := \frac{p(x)}{p'(x)}$$

is concave on $K_{p',d}$.

Proof. We proceed by evaluating the Hessian of $q_n(x)$. Note that

$$q_n(x) = \frac{p(x)}{p'(x)} = \frac{1}{\frac{1}{x_1} + \cdots + \frac{1}{x_n}}$$

so

$$\frac{\partial q_n}{\partial x_i}(x) = \frac{\left(\frac{1}{x_i}\right)^2}{\left(\frac{1}{x_1} + \cdots + \frac{1}{x_n}\right)^2}, \quad i = 1, \ldots, n$$

$$\frac{\partial^2 q_n(x)}{\partial x_i \partial x_j} = \frac{2\left(\frac{1}{x_i}\right)^2\left(\frac{1}{x_j}\right)^2}{\left(\frac{1}{x_1} + \cdots + \frac{1}{x_n}\right)^3}, \quad i \neq j$$

$$\frac{\partial^2 q_n(x)}{\partial x_i \partial x_i} = \frac{2\left(\frac{1}{x_i}\right)^2\left(\frac{1}{x_i}\right)^2}{\left(\frac{1}{x_1}+\cdots+\frac{1}{x_n}\right)^4}\left(\left(\frac{1}{x_1}+\cdots+\frac{1}{x_n}\right)-\frac{\left(\frac{1}{x_1}+\cdots+\frac{1}{x_n}\right)^2}{\left(\frac{1}{x_i}\right)}\right)$$

Denoting

$$M := -\text{Diag}\left(\left(\frac{\left(\frac{1}{x_i}\right)^3}{\left(\frac{1}{x_1}+\cdots+\frac{1}{x_n}\right)^3}\right)_{i=1}^{n}\right)$$

we can rewrite the Hessian of q_n as

$$\nabla^2 q_n = 2\left(\frac{1}{x_1}+\cdots+\frac{1}{x_n}\right)\left(\nabla q_n \nabla q_n{}^T + M\right)$$

Note that $q_n(x) < 0$ in the interior of $K_{p'} \backslash K_p$, and exactly one of $x_i < 0$ while the rest of the coordinates are positive (follows from the root interlacing property mentioned before). Thus M has all diagonal entries > 0 (eigenvalues) with the exception of one. By adding a rank-one positive semi-definite matrix $\nabla q_n \nabla q_n{}^T$ to M the eigenvalues of a resulting matrix can only be shifted to the right. But note that $\nabla^2 q_n(x)x = 0$ (because q_n is homogeneous of degree 1), henceforth the smallest (i.e., the negative) eigenvalue of M becomes 0 under this rank-one perturbation. Therefore, $\nabla^2 q_n(x) \preceq 0$. For $x \in \partial K_p / \partial K_{p'}$ use the limiting argument.

In the int$K_p = \mathbb{R}^n_{++}$ one can demonstrate the concavity as follows. We need

$$q_n(\alpha x + (1-\alpha)y) \geq \alpha q_n(x) + (1-\alpha) q_n(y) \ , \ \forall \alpha \in [0,1], x, y \in \mathbb{R}^n_{++}$$

Note that $\frac{1}{x}$ is convex on $\mathbb{R}_{++}$, from which follows that

$$\begin{aligned} q_n(\alpha x + (1-\alpha)y) &\geq \frac{1}{\frac{\alpha}{x_1}+\frac{(1-\alpha)}{y_1}+\cdots+\frac{\alpha}{x_n}+\frac{(1-\alpha)}{y_n}} \\ &= \frac{1}{\alpha\left(\frac{1}{x_1}+\cdots+\frac{1}{x_n}\right)+(1-\alpha)\left(\frac{1}{y_1}+\cdots+\frac{1}{y_n}\right)} \\ &\geq \frac{\alpha}{\frac{1}{x_1}+\cdots+\frac{1}{x_n}}+\frac{(1-\alpha)}{\frac{1}{y_1}+\cdots+\frac{1}{y_n}} = \alpha q_n(x) + (1-\alpha) q_n(y) \end{aligned}$$

□

Remark 5.2. Note that if L is an affine space not containing the origin then $q_n(x)$ is strictly concave on the relative interior of $(K_{p'} \setminus K_p) \bigcap L)$ since $\nabla^2 q_n(x) \prec 0$ on this set because we eliminate the only possible direction of singularity for this matrix.

Theorem 5.3. *Assume* $0 \leq k \leq (n-1)$. *Let* $d \in \mathbb{R}^n_{++}$, $p(x) = E_n^{(k)}(x)$, $p' = E_n^{(k+1)}(x)$ *(with respect to d). Then*

$$q_d(x) := \frac{p(x)}{p'(x)}$$

is concave on $K_{p',d}$.

Proof. Recall $p(x) = p(d)\prod_{j=1}^{m}\lambda_j(x)$ and $p'(x) = p(d)\sum_{j=1}^{m}\prod_{k\neq j}\lambda_k(x)$ where $\lambda(x)$ is the vector of eigenvalues of x with respect to p, d. We can write

$$q_d(x) = \frac{p(x)}{p'(x)} = \Psi(\Phi(x))$$

where $\Psi(y) = \frac{E_m}{E_{m-1}}(A^{-1}y) : \mathbb{R}^m \to \mathbb{R}^m$ with A satisfying

$$A_{ij} = \begin{cases} 1, & j \leq i \\ 0, & j > i \end{cases}$$

and $\Phi(x) : x \mapsto A\lambda(x) = s(x)$ (maps x onto the sums of smallest eigenvalues, $s(x)$).

If $\lambda(x)$ (and consequently $\Phi(x)$) is differentiable at a point $x \in \text{int}K_{p'}$, we can express the gradient and the Hessian of $q_d(x)$ as follows:

$$\nabla q_d(x) = \left(\frac{\partial\Psi(\Phi(x))}{\partial x_i}\right)_{i=1}^{n} = \left(\sum_{j=1}^{m}\frac{\partial\Psi}{\partial\Phi_j}\frac{\partial\Phi_j}{\partial x_i}\right)_{i=1}^{n} = \Phi'(x)^T\nabla\Psi(\Phi(x))$$

$$\nabla^2 q_d = (\Phi')^T\nabla^2\Psi\Phi' + \sum_{k=1}^{n}\frac{\partial\Psi}{\partial\Phi_k}\nabla^2\Phi_k$$

where

$$(\Phi'(x))_{ij} = \frac{\partial\Phi_i}{\partial x_j}, \quad \text{for} \quad i, j = 1, \ldots, n$$

Even though $\lambda(x)$ is not differentiable everywhere, it is differentiable at all x such that the components of $\lambda(x)$, $\lambda_i(x)$, are distinct (see, for example, [1]). For $p(x) = E_n^{(k)}(x)$ (with respect to hyperbolicity direction d), if x is not a such point, i.e., if $\exists m > 1$ such that $\lambda_j = \cdots = \lambda_{j+m}$ for some j, then by the root interlacing property it must be that $x_j/d_j = \cdots = x_{j+k+m}/d_{j+k+m}$, since these are the roots of $t \mapsto E_n(x - td)$. So we can easily choose a different hyperbolicity direction $\tilde{d} \in B(d, \epsilon) \subset \mathbb{R}^n_{++}$ (in a small ball of radius ϵ around d) such that the roots of $t \mapsto E_n(x - t\tilde{d})$ (i.e., $x_i/\tilde{d}_i$) will be distinct, and so will be the roots of $t \mapsto E_n^{(k)}(x - t\tilde{d})$. Since $q_d(x)$ is differentiable $\forall x \in \text{int}K_{p'}$, we can use the limiting argument letting $\epsilon \downarrow 0$ (so that $\nabla q_{\tilde{d}}(x) \to \nabla q_d(x)$ and $\nabla^2 q_{\tilde{d}}(x) \to \nabla^2 q_d(x)$ as $\tilde{d} \to d$). If x is a point where $\lambda(x)$ is non-differentiable, the equality above should be taken in the limiting sense as just described.

For the Hessian, the first term in the expression above is negative semi-definite (since $\nabla^2\Psi(y) \preceq 0$, $\forall y \in AC(E_{m-1}, \mathbf{1})$, see Proposition 5.1 above). Also, $\nabla^2\Phi_k(x) \preceq 0$ since the functions $\Phi(x)$ are concave (see Fact 2.2). We show that $0 \leq \frac{\partial\Psi}{\partial s_k} = \frac{\partial\Psi}{\partial\Phi_k}$, $\forall k$.

Recall that

$$\frac{\partial\left(\frac{E_m}{E_{m-1}}(\lambda)\right)}{\partial\lambda_i} = \frac{-1}{(\lambda_i)^2}\frac{-1}{\left(\frac{1}{\lambda_1} + \cdots + \frac{1}{\lambda_m}\right)^2} \geq 0$$

and $s = A\lambda$, A – invertible. Then for $i = m$

$$\frac{\partial\Psi}{\partial s_m} = \frac{\partial(s_m - s_{m-1})}{\partial s_m}\frac{\partial\left(\frac{E_m}{E_{m-1}}(\lambda)\right)}{\partial\lambda_m} = \frac{(1/\lambda_m)^2}{(1/\lambda_1 + \cdots + 1/\lambda_m)^2} \geq 0$$

and for $i < m$

$$\begin{aligned}\frac{\partial\Psi}{\partial s_i} &= \frac{\partial(s_i - s_{i-1})}{\partial s_i}\frac{\partial\left(\frac{E_m}{E_{m-1}}(\lambda)\right)}{\partial\lambda_i} + \frac{\partial(s_{i+1} - s_i)}{\partial s_i}\frac{\partial\left(\frac{E_m}{E_{m-1}}(\lambda)\right)}{\partial\lambda_{i+1}} \\ &= \frac{1}{(\lambda_i)^2}\frac{1}{\left(\frac{1}{\lambda_1} + \cdots + \frac{1}{\lambda_m}\right)^2} - \frac{1}{(\lambda_{i+1})^2}\frac{1}{\left(\frac{1}{\lambda_1} + \cdots + \frac{1}{\lambda_m}\right)^2}\end{aligned}$$

Since on K_p we have $0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m$, it follows that $\frac{\partial\Psi}{\partial s_i} \geq 0, \forall i$, for $x \in K_p$. On $K_{p'}$ we have $0 \leq \lambda_2 \leq \cdots \leq \lambda_m$ and thus $\frac{\partial\Psi}{\partial s_i} \geq 0, \forall i \geq 2$. On $K_{p'}\backslash K_p$ we have $\lambda_1 \leq 0 \leq \lambda_2 \leq \cdots \leq \lambda_m$, but we also must have $|\lambda_1| \leq |\lambda_2|$ (see the following lemma and the corollary). Therefore, $\frac{\partial\Psi}{\partial s_i} \geq 0, \forall i$, for any x in $K_{p'}$. □

Lemma 5.4. *Let $p(t)$ be an arbitrary polynomial over $\mathbb{R}$ of degree m with all real roots and let p' be its derivative polynomial. Let λ and ξ be the roots of p and p' such that $\lambda_1 \leq \xi_1 \leq \lambda_2 \leq \xi_2 \leq \cdots \leq \xi_{m-1} \leq \lambda_m$ (i.e., $p(t) = \alpha\prod_{i=1}^m(t-\lambda_i), p'(t) = \alpha\sum_{i=1}^m\prod_{j\neq i}(t-\lambda_i) = \alpha\prod_{k=1}^{m-1}(t-\xi_k)$). Then ξ_i satisfies*

$$\xi_i \in \left[\lambda_i + \frac{(\lambda_{i+1} - \lambda_i)}{m - i + 1}, \lambda_{i+1} - \frac{(\lambda_{i+1} - \lambda_i)}{i + 1}\right], \quad \forall i$$

Proof. If for some i, $\lambda_i = \lambda_{i+1}$, then $\xi_i = \lambda_i$ as well, and the statement follows trivially. Furthermore, by the root interlacing property it follows that if $\lambda_i < \lambda_{i+1}$, then $\lambda_i < \xi_i < \lambda_{i+1}$, which is the case we consider next.

Consider the ratio $\frac{p'}{p}(t)$ on the real line except for the points where $p(t) = 0$. Observe that $\frac{p'(t)}{p(t)} = 0$ iff $p'(t) = 0$ on this set. But

$$\frac{p'(t)}{p(t)} = \frac{\sum_{i=1}^m\prod_{j\neq i}(t - \lambda_i)}{\prod_{i=1}^m(t - \lambda_i)} = -\left(\frac{1}{\lambda_1 - t} + \cdots + \frac{1}{\lambda_m - t}\right)$$

hence

$$\frac{1}{\lambda_1 - \xi_i} + \frac{1}{\lambda_2 - \xi_i} + \cdots + \frac{1}{\lambda_m - \xi_i} = 0$$

Note that from the root ordering we have

$$-\frac{1}{\lambda_i - \xi_i} \leq \left(\frac{1}{\lambda_{i+1} - \xi_i} + \cdots + \frac{1}{\lambda_m - \xi_i}\right) \leq \frac{m - i}{\lambda_{i+1} - \xi_i}$$

and

$$-\frac{i}{\lambda_i - \xi_i} \geq \left(\frac{1}{\lambda_{i+1} - \xi_i} + \cdots + \frac{1}{\lambda_m - \xi_i}\right) \geq \frac{1}{\lambda_{i+1} - \xi_i}$$

The conclusion follows. □

Corollary 5.5. *Let p and p' be an arbitrary hyperbolic polynomial (of degree m) and its derivative (w.r.t. d). Let $K_p, K_{p'}$ be the corresponding cones, let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m) \in \mathbb{R}^m$ with $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_m$ be the roots of $t \mapsto p(x - td)$. Then $|\lambda_1| \leq |\lambda_2|$ on $K_{p'} \backslash K_p$.*

Proof. Denote the roots of p' by $\xi_1 \leq \xi_2 \leq \ldots \leq \xi_{m-1}$. From the previous lemma we have that $\xi_1 \leq \lambda_2 - \frac{(\lambda_2 - \lambda_1)}{2}$. But we know that $\xi_1 \geq 0$ (since we are in $K_{p'}$). Therefore, $\frac{\lambda_2}{2} + \frac{\lambda_1}{2} \geq \xi_1 \geq 0$ and $\lambda_2 \geq -\lambda_1$ and since $\lambda_1 \leq 0$ on $K_{p'} \backslash K_p$ we are done. □

The corollary completes the proof of Theorem 5.3.

Remark 5.6 (On generalization of Theorem 5.3). In order to adapt the proof of Theorem 5.3 to the case of an arbitrary hyperbolic polynomial p and its derivative polynomial p' (with respect to some d), a finer justification for differentiation by parts is needed.

6 On the structure of the associated hyperbolicity cones and their dual cones

Firstly, we note that the cone $K_{E_k,\mathbf{1}}$ has a recursive structure similar to $\mathbb{R}^n_+ = K_{E_n,\mathbf{1}}$: by dropping some of the coordinates of $x \in K_{E_k,\mathbf{1}}$, we obtain a vector in "almost the same" cone (with respect to the degree of the underlying polynomial) but in a lower dimensional space (compare with: a face of a simplex is a simplex). In turn, this gives us a different characterization for $K_{E_k,\mathbf{1}}$.

Following this observation to gain insight into the dual cones $K^*_{E_k,\mathbf{1}}$, we create a suitable decomposition of the cone $K_{E_{n-1},\mathbf{1}}$ into smaller convex cones, suitable in the sense that each of the smaller cones admits a positive semidefinite representation (see the definition 4.1). Relying on the conic duality theory, we then obtain the dual cone for each of the smaller cones as an SDR set in itself, and finally, we reconstruct $K^*_{E_{n-1},\mathbf{1}}$ as the intersection.

For a vector $x \in \mathbb{R}^n$ and an arbitrary index $1 \leq i \leq n$, we write $x_{-i} \in \mathbb{R}^{n-1}$ for a vector whose i^{th} coordinate has been removed.

Throughout this section fix the underlying vector space to be $\mathbb{R}^n$, the hyperbolicity direction $d \equiv \mathbf{1}$. For a fixed $2 \leq k \leq (n-1)$, denote $p(t) : t \mapsto E_k(x + t\mathbf{1})$, for $1 \leq i \leq n$ denote $p_{-i}(t) : t \mapsto E_k(x_{-i} + t\mathbf{1}_{-i})$ and $p'_{-i}(t) : t \mapsto (n-k)E_{k-1}(x_{-i} + t\mathbf{1}_{-i})$.

6.1 The recursive structure of the hyperbolicity cones for elementary symmetric polynomials

Observe the recursive expression $E_k(x) = x_i E_{k-1}(x_{-i}) + E_k(x_{-i})$ for any $n > k \geq 2$ and an arbitrary index i, where $E_k(\cdot_{-i}) : \mathbb{R}^{n-1} \to \mathbb{R}$ is the k^{th} elementary symmetric polynomial on $\mathbb{R}^{n-1}$.

Theorem 6.1 (Necessary condition for $x \in K_{E_k}$). *Assume $2 \le k \le n$. Then $x \in K_{E_k(\cdot)}$ only if $x_{-i} \in K_{E_{k-1}(\cdot_{-i})}$, $\forall i$.*

Proof. If $k = n$ the result is obvious, so assume $k < n$. Fix i. Write $E_k(x) = x_i E_{k-1}(x_{-i}) + E_k(x_{-i})$ and recall that $x \in K_{E_k(\cdot)}$ iff $p(t) : t \mapsto E_k(x + t\mathbf{1})$ has only non-positive roots. $E_k(\cdot_{-i})$ and $E_{k-1}(\cdot_{-i})$ are both hyperbolic along $\mathbf{1}_{-i} \in \mathbb{R}^{n-1}$,

$$\lim_{t\uparrow\infty} \frac{E_{k-1}(x_{-i}+t\mathbf{1}_{-i})}{t^{k-1}} \ge 0 \quad \text{and} \quad \lim_{t\uparrow\infty} \frac{E_k(x_{-i}+t\mathbf{1}_{-i})}{t^k} \ge 0,$$

$\forall x \in \mathbb{R}^n$ (as $t \uparrow \infty$, $E_{k-1}(x_{-i} + t\mathbf{1}_{-i})$ and $E_k(x_{-i} + t\mathbf{1}_{-i})$ will eventually be ≥ 0).

Using $p_{-i}(t), p'_{-i}(t)$ as defined previously we can write $p(t) = E_k(x + t\mathbf{1}) = \frac{(x_i+t)}{n-k} p'_{-i}(t) + p_{-i}(t)$.

Suppose $x_{-i} \notin K_{E_{k-1}(\cdot_{-i})}$, so there must be at least one positive root of $p'_{-i}(t)$. We also know that roots of $p_{-i}(t)$ and $p'_{-i}(t)$ are interlaced: enumerating all roots (including multiplicities) of $p_{-i}(t)$ as $\{t_i : i = 1, \dots, k\}$ and roots of $p'_{-i}(t)$ as $\{t'_i : i = 1, \dots, (k-1)\}$ in non-decreasing order we must have $t_1 \le t'_1 \le t_2 \le t'_2 \le \cdots \le t_{k-1} \le t'_{k-1} \le t_k$, $0 < t'_{k-1} \le t_k$ and also from the observation made about signs of $p_{-i}(t)$ and $p'_{-i}(t)$ as $t \uparrow \infty$ we get that

$$\begin{gathered} p'_{-i}(t) \ge 0 \quad \text{for} \quad t \ge t'_{k-1}, \\ p_{-i}(t'_{k-1}) \le 0 \quad \text{and} \quad p_{-i}(t) \ge 0 \quad \text{for} \quad t \ge t_k \end{gathered}$$

We consider three cases depending on the value x_i.

Case 1. Suppose that $-x_i \le t'_{k-1}$. Then

$$\begin{aligned} p(t'_{k-1}) &= \tfrac{(x_i+t'_{k-1})}{n-k} p'_{-i}(t'_{k-1}) + p_{-i}(t'_{k-1}) \le 0 \\ p(t_k) &= \tfrac{(x_i+t_k)}{n-k} p'_{-i}(t_k) + p_{-i}(t_k) \ge 0 \end{aligned}$$

so by continuity, $p(t)$ must have a root between t'_{k-1} and t_k. Since $0 \le t'_{k-1}$, this root must be positive, hence $x \notin K_{E_k(\cdot)}$ (see Figure 1).

Case 2. Suppose $t'_{k-1} < x_i \le t_k$. Then we can write

$$\begin{aligned} p(-x_i) &= \tfrac{(x_i+(-x_i))}{n-k} p'_{-i}(-x_i) + p_{-i}(-x_i) \le 0 \\ p(t_k) &= \tfrac{(x_i+t_k)}{n-k} p'_{-i}(t_k) + p_{-i}(t_k) \ge 0 \end{aligned}$$

and again by continuity, $p(t)$ must have a positive root, so $x \notin K_{E_k(\cdot)}$.

Case 3. Finally, suppose that $t_k < -x_i$. Then

$$\begin{aligned} p(t_k) &= \tfrac{(x_i+t_k)}{n-k} p'_{-i}(t_k) + p_{-i}(t_k) \le 0 \\ p(-x_i) &= \tfrac{(x_i+(-x_i))}{n-k} p'_{-i}(-x_i) + p_{-i}(-x_i) \ge 0 \end{aligned}$$

so by continuity, $p(t)$ must have a positive root and therefore $x \notin K_{E_k(\cdot)}$. □

Corollary 6.2. *Assume $2 \le k \le (n-1)$ and $x \in K_{E_k(\cdot)}$. If $x_i \le 0$ then $x_{-i} \in K_{E_k(\cdot_{-i})}$. Moreover, if $x \in \partial K_{E_k(\cdot)}$, $x \notin \mathbb{R}^n_+$, and $x_i > 0$, then $x_{-i} \notin K_{E_k(\cdot_{-i})}$.*

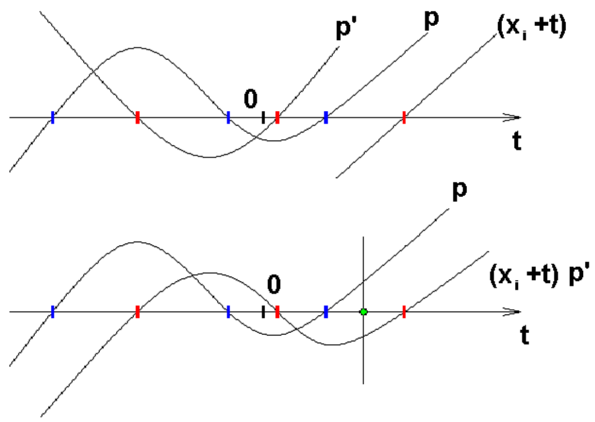

Figure 1: Necessary condition for $x \in K_{E_k}$, root interlacing case 1

Proof. We write $p(t) = E_k(x + t\mathbf{1}) = \frac{(x_i+t)}{n-k} p'_{-i}(t) + p_{-i}(t)$ and at $t = 0$ we have $E_k(x) = p(0) = \frac{x_i}{n-k} p'_{-i}(0) + p_{-i}(0) = x_i E_{k-1}(x_{-i}) + E_k(x_{-i})$. Since $x_{-i} \in K_{E_{k-1}(\cdot_{-i})}$ by Theorem 6.1, we have $p'_{-i}(0) = (n-k)E_k(x_{-i}) \geq 0$, and also from Theorem 3.2, $p(0) = E_k(x) \geq 0$. We rearrange terms: $p'_{-i}(0)x_i = (n-k)(E_k(x) - p_{-i}(0))$.

If $x_i \leq 0$ we have $E_k(x) - p_{-i}(0) \leq 0$, so $p_{-i}(0) = E_k(x_{-i}) \geq 0$ and combined with $x_{-i} \in K_{E_{k-1}(\cdot_{-i})}$, this gives us $x_{-i} \in K_{E_k(\cdot_{-i})}$.

Now let $x \in \partial K_{E_k(\cdot)}$, so that $E_k(x) = 0$, and $x_i > 0$. We have two possibilities here. If $p'_{-i}(0) > 0$, then $-p_{-i}(0) > 0$ and hence $x_{-i} \notin K_{E_k(\cdot_{-i})}$. Alternatively, if $p'_{-i}(0) = 0$ ($x \in \partial K_{E_{k-1}(\cdot_{-i})}$), then $p_{-i}(0) = 0$ and $x_{-i} \in \partial K_{E_k(\cdot_{-i})}$, so by Proposition 3.5, $x \in K_{E_{n-1}(\cdot_{-i})} \equiv \mathbb{R}^{n-1}_+$. But $x \notin \mathbb{R}^n_+$, we have a contradiction. ☐

6.2 Alternative characterization of the hyperbolicity cones associated with elementary symmetric polynomials

Instead of considering $x \in \mathbb{R}^n$ we confine ourselves to the cone $\mathbb{R}^n_\downarrow = \{x \in \mathbb{R}^n : x_n \leq x_{n-1} \leq x_{n-2} \leq \cdots \leq x_1\}$.

Theorem 6.3 (K_{E_k} characterization)**.** *Assume $2 \leq k \leq n$ and $x \in \mathbb{R}^n_\downarrow$. Then $x \in K_{E_k(\cdot)}$ iff $x_{-n} \in K_{E_{k-1}(\cdot_{-n})}$ and $E_k(x) \geq 0$.*

Proof. The conditions are necessary (see previous lemma and Theorem 3.2). We need to show sufficiency. The case $k = n$ is trivial, so assume $k < n$. If $x_n \geq 0$, then obviously $x \in K_{E_k(\cdot)} (\supset \mathbb{R}^n_+)$, so assume $x_n < 0$.

Let $p(t)$, $p_{-n}(t)$ and $p'_{-n}(t)$ with corresponding roots (including multiplicities) of $p_{-n}(t)$, $\{t_i : i = 1, \ldots, k\}$, and roots of $p'_{-n}(t)$, $\{t'_i : i = 1, \ldots, (k-1)\}$, in non-decreasing order be as before. Write $p(t) = E_k(x + t\mathbf{1}) = \frac{(x_n+t)}{n-k} p'_{-n}(t) + p_{-n}(t))$.

Observe

$$p(t_k) = \frac{(x_n + t_k)}{n-k} p'_{-n}(t_k) \le 0$$

since $(x_n + t_k) \le 0$ (by the root interlacing $t_k \le -x_n$) and $p'_{-n}(t_k) \ge 0$ (recall that $p'_{-n}(t) \uparrow \infty$ as $t \uparrow \infty$). Also, $p(0) = E_k(x) \ge 0$ by the assumption. Thus the interval $[t_k, 0]$ must contain at least one root of $p(t)$.

By counting the remaining roots of $p(t)$, $t \le t_k$ (by looking at sign patterns at the endpoints of intervals $[t_i, t'_i], i = 1, \ldots, (k-1))$ we conclude that $[t_k, 0]$ must contain only one (rightmost) root of $p(t)$ (so there could be no other roots to the right of 0) and hence $x \in K_{E_k(\cdot)}$. □

Corollary 6.4. *Assume* $2 \le k \le (n-1)$ *and* $x \in \mathbb{R}^n_\downarrow$. *Then* $x \in K_{E_k(\cdot)}$ *iff* $x_{-n} \in K_{E_k(\cdot_{-n})}$ *and* $E_k(x) \ge 0$.

Proof. Straightforward. □

We make one observation about $\mathbb{R}^n_\downarrow$, namely, for an arbitrary index i and any $k \ge 0$ we have $x_{-i} \le x_{-(i+k)}$ (easy to check).

Let $\tilde{q}(\cdot_{-i}) := \frac{E_k(\cdot_{-i})}{E_{k-1}(\cdot_{-i})} = (n-k)\frac{p_{-i}(0)}{p'_{-i}(0)}$ and recall that this function was shown to be concave on $K_{E_{k-1}(\cdot_{-i})}$ and is 1-homogeneous.

Proposition 6.5. *Assume* $2 \le k \le n$. *Let* $\tilde{q}(\cdot) := \frac{E_k(\cdot)}{E_{k-1}(\cdot)}$. *If* $x, y \in K_{E_{k-1}(\cdot)}$, *then* $\tilde{q}(x+y) \ge \tilde{q}(x) + \tilde{q}(y)$.

Proof. Since $\tilde{q}(x)$ is concave (see Theorem 5.3) we can write $-\tilde{q}(\frac{x+y}{2}) \le \frac{-\tilde{q}(x) - \tilde{q}(y)}{2}$ and from homogeneity it follows that $-\tilde{q}(x+y) \le -\tilde{q}(x) - \tilde{q}(y)$. □

Remark 6.6 (On a set of necessary conditions, compare with Corollary 6.2). Assume $2 \le k \le (n-1)$ and $x \in \mathbb{R}^n_\downarrow$. If $x \in \partial K_{E_k(\cdot)}$, then $\exists j$ such that

$$\begin{array}{ll} x_{-i} \in K_{E_k(\cdot_{-i})} & \text{for} \quad i \ge j \\ x_{-i} \notin K_{E_k(\cdot_{-i})} & \text{for} \quad i < j \end{array}$$

Proof. For fixed i and $k \ge 0$, $x_{-i} \le x_{-(i+k)}$. Observe $\tilde{q}(x_{-(i+k)}) = \tilde{q}(x_{-i} + (x_{-(i+k)} - x_{-i})) \ge \tilde{q}(x_{-i}) + \tilde{q}(x_{-(i+k)} - x_{-i}) \ge \tilde{q}(x_{-i})$ since $(x_{-(i+k)} - x_{-i}) \in \mathbb{R}^n_+$. The condition on x_{-i} being in or out of $K_{E_k(\cdot_{-i})}$ for $i < n$ (i.e., $\tilde{q}(\cdot_{-i})$ having the right sign) is implied by "monotonicity" of $\tilde{q}(\cdot_{-i})$. □

6.3 First derivative cone for $\mathbb{R}^n_+$ and its dual

Recall for any hyperbolic polynomial h one can give a characterization of the (closure of) associated hyperbolicity cone K_h by the set of polynomial inequalities as in Theorem 3.2. For $p(x) = E_n(x)$ and its derivative $p'(x) = E_{n-1}(x)$ (w.r.t. $d = \mathbf{1}$) we claim that $x \in K_{p'}$ iff $p'(x) \ge 0$ and at most one $x_i < 0$ with the rest $x_j \ge 0$ for $i \ne j$ (follows from Corollary 6.4). We are going to construct a representation of the dual cone to $K_{p'} = K_{E_{n-1}}$ using this characterization.

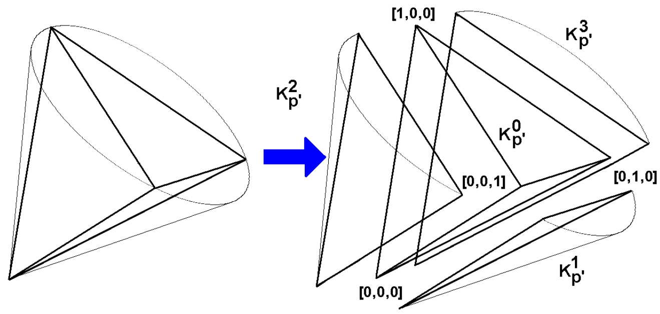

Figure 2: $K_{E_{n-1}}$ cone decomposition in $\mathbb{R}^3$

Proposition 6.7. *If $K \subseteq \mathbb{R}^n$ is a cone admitting a decomposition into (smaller) cones $\{K_i\}_{i\in I}$, $K = \bigcup_{i\in I} K_i$, then its dual cone satisfies $K^* = \bigcap_{i\in I} {K_i}^*$.*

Proof. Straightforward from the definition of the dual cone. □

We form a (disjoint-interior) partitioning for $K_{p'}$ in the following manner: $K_{p'} = (\bigcup_{i=1\dots n} K^i_{p'}) \bigcup K^0_{p'}$ where $K^i_{p'} = \{x \in \mathbb{R}^n : x_i \leq 0, x_j \geq 0, j \neq i, p'(x) \geq 0\}$ and $K^0_{p'} = K_p = (K_p)^* = \mathbb{R}^n_+$, claiming that each of the $K^i_{p'}$ admits SDR representation (with strictly-feasible solution), see Figure 2. Based on Proposition 4.3 it is now easy to reconstruct the dual cone.

It is left to demonstrate how to represent each $K^i_{p'}$ via linear matrix inequality (LMI). We show how to do this for $K^1_{p'}$.

Consider

$$W_1(x) := \mathrm{Diag}(x) - x_1(\mathbf{1}\cdot(1,0,\dots,0)^T + (1,0,\dots,0)\cdot\mathbf{1}^T)$$

The condition of the form $W_1(x) \succeq 0$ is clearly an LMI. Recall that for a real symmetric matrix to be positive semi-definite it is necessary and sufficient that all its principal minors have nonnegative determinants. Proceed by evaluating these determinants from the bottom-right corner to get $x_j, j = 2,\dots,n$, $x_1 \leq 0$ and $-x_1 p'(x) \geq 0$, implying $p'(x) \geq 0$. (To see why $\det(W_1(x)) = -x_1 p'(x)$, evaluate this determinant using algebraic complements of the first row, see [9].)

Clearly, strict feasibility for this LMI is insured as well (e.g., take $x_2 = x_3 = \dots = x_n = 1, x_1 < 0$, with $|x_1|$ small enough). So Corollary 4.3 can be applied to get $(K^1_{p'})^* = \{x \in \mathbb{R}^n : W_1(x) \succeq 0\}^*$ as an SDR set.

Finally, to get the representation of the dual cone to $K_{E_{n-1}(\cdot)}$ take the intersection of the dual cones corresponding to its components: $(K_{E_{n-1}(\cdot)})^* = (\cap_{i=1,\dots,n}\{x \in \mathbb{R}^n : W_i(x) \succeq 0\}^*) \cap (\mathbb{R}^n_+)^*$.

To illustrate this idea we consider the derivation of $(K_{E_{n-1}(\cdot)})^*$ in $\mathbb{R}^3$, which is, perhaps, not the most exciting example (it is just a quadratic cone after all) but is quite an illustrative one (it is easy to appeal to geometric interpretation of the results).

The dual cone is given by $(\cap_{i=1,\ldots,n}\{x \in \mathbb{R}^n : W_i(x) \succeq 0\}^*) \cap (\mathbb{R}^n_+)^*$. Consider $\{x \in \mathbb{R}^n : W_1(x) \succeq 0\}$ first:

$$W_1(x) : \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto \begin{bmatrix} -1 & -1 & -1 \\ -1 & & \\ -1 & & \end{bmatrix} x_1 + \begin{bmatrix} 0 & & \\ & 1 & \\ & & 0 \end{bmatrix} x_2 + \begin{bmatrix} 0 & & \\ & 0 & \\ & & 1 \end{bmatrix} x_3$$
$$= A_1 x_1 + A_2 x_2 + A_3 x_3$$

Using Proposition 4.3 we get the following representation of $\{x \in \mathbb{R}^n : W_i(x) \succeq 0\}^*$:

$$\langle \begin{bmatrix} -1 & -1 & -1 \\ -1 & & \\ -1 & & \end{bmatrix}, \Lambda_1 \rangle = y_1, \quad \langle \begin{bmatrix} 0 & & \\ & 1 & \\ & & 0 \end{bmatrix}, \Lambda_1 \rangle = y_2, \quad \langle \begin{bmatrix} 0 & & \\ & 0 & \\ & & 1 \end{bmatrix}, \Lambda_1 \rangle = y_3,$$
$$\Lambda_1 \succeq 0$$

and similarly we can derive the expressions for $\{x \in \mathbb{R}^n : W_2(x) \succeq 0\}^*$ (with Λ_2) and $\{x \in \mathbb{R}^n : W_3(x) \succeq 0\}^*$ (with Λ_3). We reconstruct the dual cone to $K_{E_2(\cdot)}$ as a collection of three sets of LMI's each corresponding to $\{x \in \mathbb{R}^n : W_i(x) \succeq 0\}^*$ (with the same y in all of them, $i = 1, 2, 3$). Note that there is no need to further restrict ourselves to $y \in \mathbb{R}^3_+$ since this is already implied by the constraints.

An interesting question that remains unanswered is: "How would one get the (complete) representation of the original cone $K_{E_2(\cdot)}$ in terms of LMI's ?". To do this we take the dual of $K^*_{E_{n-1}(\cdot)}$. Firstly, let us switch from the image of a positive semi-definite cone to its affine slice in each of the $\{x \in \mathbb{R}^n : W_i(x) \succeq 0\}^*$, $i = 1, \ldots, n$. Starting with $\{x \in \mathbb{R}^n : W_1(x) \succeq 0\}^*$, fixing a basis in $\mathbb{S}^3$ to be

$$\{B_i\}_{i=1}^6 = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$$

we can write (substituting $\Lambda_1 = \sum_{j=1}^6 B_j \lambda_{1j}$) $\langle A_i, \Lambda_1 \rangle = y_i \Leftrightarrow \langle A_i, \sum_{j=1}^6 B_j \lambda_{1j} \rangle = y_i \Leftrightarrow \sum_{j=1}^6 \langle B_j, A_i \rangle \lambda_{1j} = y_i$ to get

$$\begin{bmatrix} -y_1 - 2(\lambda_{12} + \lambda_{13}) & \lambda_{12} & \lambda_{13} \\ \lambda_{12} & y_2 & \lambda_{15} \\ \lambda_{13} & \lambda_{15} & y_3 \end{bmatrix} \succeq 0$$

for $y \in \left(K^1_{p'}\right)^*$, and similarly for $\{x \in \mathbb{R}^3 : W_2(x) \succeq 0\}^*$, $\{x \in \mathbb{R}^3 : W_2(x) \succeq 0\}^*$.

Now we can apply the same procedure to take the dual of the dual cone to get the primal cone itself (note again that the resulting LMI is strictly feasible, for example, take $y = \mathbf{1}, -1/2 < \lambda_{i,j} < -1/3, \forall i, j$). Writing the constraints corresponding to $K^*_{E_{n-1}}$ in the form $\sum_{i=1}^3 y_i \widetilde{A}_i + \sum_{j=1}^3 (\lambda_{j2} \widetilde{B}_{j2} + \lambda_{j3} \widetilde{B}_{j3} + \lambda_{j5} \widetilde{B}_{j5}) \succeq 0$, we obtain

$$\left.\begin{array}{|ccc|ccc|ccc|}
\hline
\mu_{1,11} & \mu_{1,11} & \mu_{1,11} & & & & & & \\
\mu_{1,11} & \mu_{1,22} & & & & & & & \\
\mu_{1,11} & & \mu_{1,33} & & & & & & \\
\hline
 & & & \mu_{2,11} & \mu_{2,11} & \mu_{2,11} & & & \\
 & & & \mu_{2,11} & \mu_{2,22} & & & & \\
 & & & \mu_{2,11} & & \mu_{2,33} & & & \\
\hline
 & & & & & & \mu_{3,11} & \mu_{3,11} & \mu_{3,11} \\
 & & & & & & \mu_{3,11} & \mu_{3,22} & \\
 & & & & & & \mu_{3,11} & & \mu_{3,33} \\
\hline
\end{array}\right. \succeq 0$$

$$\begin{aligned}
x_1 &= \mu_{1,33} + \mu_{2,33} - \mu_{3,11} \\
x_2 &= \mu_{1,22} - \mu_{2,11} + \mu_{3,22} \\
x_3 &= -\mu_{1,11} + \mu_{2,22} + \mu_{3,33}
\end{aligned}$$

for $x \in K_{E_2}$, where the off-diagonal blocks are not necessarily zeroes anymore, but w.lo.g. may be assumed 0 (use the criterion for a matrix to be $\succeq 0$ using the minors).

This constraint is decomposable into three independent LMI's (corresponding to these three blocks) which are further assembled together via affine constraints in order to get the primal variables x_1, x_2, x_3. There is a simple interpretation for this set of constraints. Observe that each of the blocks ($i = 1, 2, 3$)

$$\begin{bmatrix}
\mu_{i,11} & \mu_{i,11} & \mu_{i,11} \\
\mu_{i,11} & \mu_{i,22} & \\
\mu_{i,11} & & \mu_{i,33}
\end{bmatrix} \succeq 0$$

corresponds to $K^i_{p'} = \{x \in \mathbb{R}^n : x_i \leq 0, x_j \geq 0, j \neq i, E_{n-1}(x) \geq 0\} = \{x \in \mathbb{R}^n : W_i(x) \succeq 0\}$ but with x's now renamed into $\pm\mu$'s. Therefore, each block describes just one of these "slabs". The remaining linear constraints

$$\begin{aligned}
x_1 &= \mu_{1,33} + \mu_{2,33} - \mu_{3,11} \\
x_2 &= \mu_{1,22} - \mu_{2,11} + \mu_{3,22} \\
x_3 &= -\mu_{1,11} + \mu_{2,22} + \mu_{3,33}
\end{aligned}$$

are building a convex combination of these slabs ($K_{E_{n-1}(\cdot)}$ is a cone so we can assume that the points have unit weight). That is, any point in $K_{p'}$ can be obtained as a convex combination of the points in $K^i_{p'} = \{x \in \mathbb{R}^n : x_i \leq 0, x_j \geq 0, j \neq i, E_{n-1}(x) \geq 0\} = \{x \in \mathbb{R}^n : W_i(x) \succeq 0\}$, $i = 1, \ldots, n$ (see Figure 2).

Remark 6.8 (Concluding comments). It should be noted that since the dual cone $K^*_{E_{n-1}}$ was constructed as an affine section of $S^{n^2}_+$, this approach provides us with a way to construct a self-concordant barrier for the dual cone as well.

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