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Abstract

We introduce a continuous analogue of the Hirsch conjecture and a discrete analogue of the result of Dedieu, Malajovich and Shub. We prove a continuous analogue of the result of Holt and Klee, namely, we construct a family of polytopes which attain the conjectured order of the largest total curvature.

Keywords: polytopes, arrangements, diameter, central path, total curvature.

1 Continuous analogue of the conjecture of Hirsch

By analogy with the conjecture of Hirsch, we conjecture that the order of the largest total curvature of the central path associated to a polytope is the number of inequalities defining the polytope. By analogy with a result of Dedieu, Malajovich and Shub, we conjecture that the average diameter of a bounded cell of a simple arrangement is less than the dimension. We prove a continuous analogue of the result of Holt-Klee, namely, we construct a family of polytopes which attain the conjectured order of the largest total curvature. We substantiate the conjectures in low dimensions and highlight additional links.

Let P be a full dimensional convex polyhedron defined by m inequalities in dimension n. The diameter $\delta(P)$ is the smallest number such that any two vertices of the polyhedron P can be connected by a path with at most $\delta(P)$ edges. The conjecture of Hirsch, formulated in 1957 and reported in [3], states that the diameter of a polyhedron defined by m inequalities in dimension n is not greater than $m-n$. The conjecture does not hold for unbounded polyhedra. A polytope is a bounded polyhedron. No polynomial bound is known for the diameter of a polytope.

Conjecture 1.1. (Conjecture of Hirsch for polytopes)
The diameter of a polytope defined by m inequalities in dimension n is not greater than $m-n$.

Intuitively, the total curvature [16] is a measure of how far off a certain curve is from being a straight line. Let $\psi : [\alpha, \beta] \to \mathbb{R}^n$ be a $C^2((\alpha-\varepsilon, \beta+\varepsilon))$ map for some $\varepsilon > 0$ with a non-zero derivative in $[\alpha, \beta]$. Denote its arc length by $l(t) = \int_\alpha^t \|\dot{\psi}(\tau)\| d\tau$, its parametrization by the arc length by $\psi_{\text{arc}} = \psi \circ l^{-1} : [0, l(\beta)] \to \mathbb{R}^n$, and its curvature at the point t by $\kappa(t) = \ddot{\psi}_{\text{arc}}(t)$. The total curvature is defined as $\int_0^{l(\beta)} \|\kappa(t)\| dt$. The requirement $\dot{\psi} \neq 0$ insures that any given segment of the curve is traversed only once and allows to define a curvature at any point on the curve.

From now on we consider only polytopes, i.e., bounded polyhedra, and denote those by P. For a polytope $P = \{x : Ax \geq b\}$ with $A \in \mathbb{R}^{m \times n}$, denote $\lambda(P)$ the largest total curvature of the primal central path corresponding to the standard logarithmic barrier function, $-\sum_{i=1}^{m}\ln(A_i x - b_i)$, of the linear programming problem $\min\{c^T x : x \in P\}$ over all possible c. Following the analogy with the diameter, let $\Lambda(m,n)$ be the largest total curvature $\lambda(P)$ of the primal central path over all polytopes P defined by m inequalities in dimension n.

Conjecture 1.2. (Continuous analogue of the conjecture of Hirsch)
The order of the largest total curvature of the primal central path over all polytopes defined by m inequalities in dimension n is the number of inequalities defining the polytopes, i.e., $\Lambda(m,n) = \mathcal{O}(m)$.

Remark 1.1. *In [6] the authors showed that a redundant Klee-Minty n-cube $\mathcal{C}$ satisfies $\lambda(\mathcal{C}) \geq (\frac{3}{2})^n$, providing a counterexample to the conjecture of Dedieu and Shub [5] that $\Lambda(m,n) = \mathcal{O}(n)$.*

For polytopes and arrangements, respectively central path and linear programming, we refer to the books of Grünbaum [10] and Ziegler [17], respectively Renegar [13] and Roos et al [14].

2 Discrete analogue of the result of Dedieu, Malajovich, Shub

Let $\mathcal{A}$ be a simple arrangement formed by m hyperplanes in dimension n. We recall that an arrangement is called simple if $m \geq n+1$ and any n hyperplanes intersect at a unique distinct point. Since $\mathcal{A}$ is simple, the number of bounded cells, i.e., bounded connected components of the complement to the hyperplanes, of $\mathcal{A}$ is $I = \binom{m-1}{n}$. Let $\lambda^c(P)$ denote the total curvature of the primal central path corresponding to $\min\{c^T x : x \in P\}$. Following the approach of Dedieu, Malajovich and Shub [4], let $\lambda^c(\mathcal{A})$ denote the average value of $\lambda^c(P_i)$ over the bounded cells P_i of $\mathcal{A}$; that is,

$$\lambda^c(\mathcal{A}) = \frac{\sum_{i=1}^{i=I} \lambda^c(P_i)}{I}.$$

Note that each bounded cell P_i is defined by the same number m of inequalities, some being potentially redundant. Given an arrangement $\mathcal{A}$, the average total curvature of a bounded cell $\lambda(\mathcal{A})$ is the largest value of $\lambda^c(\mathcal{A})$ over all possible c. Similarly, $\Lambda_{\mathcal{A}}(m,n)$ is the largest possible average total curvature of a bounded cell of a simple arrangement defined by m inequalities in dimension n.

Proposition 2.1. (Dedieu, Malajovich and Shub [4])
The average total curvature of a bounded cell of a simple arrangement defined by m inequalities in dimension n is not greater than $2\pi n$.

By analogy, let $\delta(\mathcal{A})$ denote the average diameter of a bounded cell of $\mathcal{A}$; that is,

$$\delta(\mathcal{A}) = \frac{\sum_{i=1}^{i=I} \delta(P_i)}{I}.$$

Similarly, let $\Delta_{\mathcal{A}}(m,n)$ denote the largest possible average diameter of a bounded cell of a simple arrangement defined by m inequalities in dimension n.

Conjecture 2.1. (Discrete analogue of the result of Dedieu, Malajovich and Shub)
The average diameter of a bounded cell of a simple arrangement defined by m inequalities in dimension n is not greater than n.

Haimovich's probabilistic analysis of the shadow-vertex simplex algorithm, see [2, Section 0.7], shows that the expected number of pivots is bounded by n. While the result and Conjecture 2.1 are similar in nature, they differ in some aspects: Haimovich considers the average over bounded and unbounded cells, and the number of pivots could be smaller than the diameter for some cells.

3 Additional links and low dimensions

3.1 Additional links

Proposition 3.1. *If the conjecture of Hirsch holds, then* $\Delta_{\mathcal{A}}(m,n) \leq n + \frac{2n}{m-1}$.

Proof. Let m_i denote the number of hyperplanes of $\mathcal{A}$ which are non-redundant for the description of a bounded cell P_i. If the conjecture of Hirsch holds, we have $\delta(P_i) \leq m_i - n$. It implies:

$$\delta(\mathcal{A}) \leq \frac{\sum_{i=1}^{i=I}(m_i - n)}{I} = \frac{\sum_{i=1}^{i=I} m_i}{I} - n.$$

Since a facet belongs to at most 2 cells, the sum of m_i for $i = 1, \ldots, I$ is less than twice the number of bounded facets of $\mathcal{A}$. As a bounded facet induced by a hyperplane H of $\mathcal{A}$ corresponds to a bounded cell of the $(n-1)$-dimensional simple arrangement $\mathcal{A} \cap H$, the sum of m_i is less than $2m\binom{m-2}{n-1}$. Therefore, we have, for any simple arrangement $\mathcal{A}$, $\delta(\mathcal{A}) \leq \frac{2m\binom{m-2}{n-1}}{\binom{m-1}{n}} - n = \frac{2mn}{m-1} - n = \frac{n(m+1)}{m-1}$. □

Remark 3.1. *In the proof of Proposition 3.1, we overestimate the sum of* m_i *for* $i = 1, \ldots, I$ *as some bounded facets belong to exactly 1 bounded cell. Let call such bounded facets external. We hypothesize that any simple arrangement has at least* $n\binom{m-2}{n-1}$ *external facets, in turn, this would strengthen Proposition 3.1 to:* If the conjecture of Hirsch holds, then $\Delta_{\mathcal{A}}(m,n) \leq \frac{n(m-n+1)}{m-1}$.

Similarly to Proposition 3.1, the results of Kalai and Kleitman [11] and Barnette [1] which bounds the diameter of a polytope by, respectively, $2\,m^{\log(n)+1}$ and $\frac{2^{n-2}}{3}(m - n + \frac{5}{2})$, directly yield:

Proposition 3.2. $\Delta_{\mathcal{A}}(m,n) \leq \frac{4mn\left(2m\binom{m-2}{n-1}\right)^{\log n}}{m-1}$ *and* $\Delta_{\mathcal{A}}(m,n) \leq n(\frac{m+1}{m-1} + \frac{5}{2n})\frac{2^{n-2}}{3}$.

The special case of $m = 2n$ of the conjecture of Hirsch is known as the d-step conjecture (as the dimension is often denoted by d in polyhedral theory). In particular, it has been shown by Klee and Walkup [12] that the special case $m = 2n$ for all n is equivalent to the conjecture of Hirsch. A continuous analogue would be: if $\Lambda(2n,n) = \mathcal{O}(n)$ for all n, then $\Lambda(m,n) = \mathcal{O}(m)$.

Remark 3.2. *In contrast with Proposition 3.1,* $\Lambda(m,n) = \mathcal{O}(m)$ *does not imply that* $\Lambda_{\mathcal{A}}(m,n) = \mathcal{O}(n)$ *since all the* m *inequalities count for each* $\lambda(P_i)$ *while it is enough to consider the* m_i *non-redundant inequalities for each* $\delta(P_i)$.

3.2 Low dimensions

In dimensions 2 and 3 we have, respectively, $\delta(P) \leq \lfloor \frac{m}{2} \rfloor$ and $\delta(P) \leq \lfloor \frac{2m}{3} \rfloor - 1$, implying:

Proposition 3.3. $\Delta_{\mathcal{A}}(m,2) \leq 2 + \frac{2}{m-1}$ *and* $\Delta_{\mathcal{A}}(m,3) \leq 3 + \frac{4}{m-1}$.

In dimension 2, let S_2 be a unit sphere centered at $(1,1)$ and consider the arrangement $\mathcal{A}^*_{m,2}$ made of the 2 lines forming the nonnegative orthant and additional $m-2$ lines tangent to S_2 and separating the origin from the center of the sphere. See Figure 1 for an illustration of $\mathcal{A}^*_{6,2}$. Besides $m-2$ triangles, the bounded cells of $\mathcal{A}^*_{m,2}$ are made of $\binom{m-2}{2}$ 4-gons. We have $\delta(\mathcal{A}^*_{m,2}) = \frac{2(m-2)}{m-1}$, and thus,

Proposition 3.4. $2 - \frac{2}{m-1} \leq \Delta_{\mathcal{A}}(m,2) \leq 2 + \frac{2}{m-1}$.

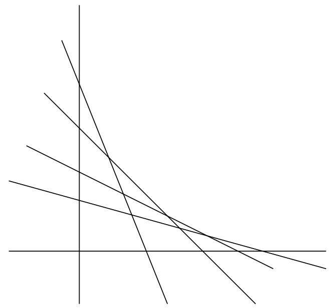

Figure 1: The arrangement $\mathcal{A}^*_{6,2}$

Remark 3.3. *The arrangement $\mathcal{A}^*_{m,2}$ was generalized in [7] to an arrangement with $\binom{m-n}{n}$ cubical cells yielding that the dimension n is an asymptotic lower bound for $\Delta_{\mathcal{A}}(m,n)$ for fixed n.*

In dimension 2, for $m \geq 4$, consider the polytope $P^*_{m,2}$ defined by the following m inequalities: $y \leq 1$, $x \leq \frac{y}{10} + \frac{1}{2}$, $-x \leq \frac{y}{3} + \frac{1}{3}$ and $(-1)^i x \leq \frac{10^{i-2}y}{11} + \frac{5}{11} - \frac{10^{-4}}{m}\frac{i}{m}$ for $i = 4, \ldots, m$. See Figure 2 for an illustration of $P^*_{6,2}$ and Figure 3 for the central path over $P^*_{34,2}$ with $c = (0,1)$.

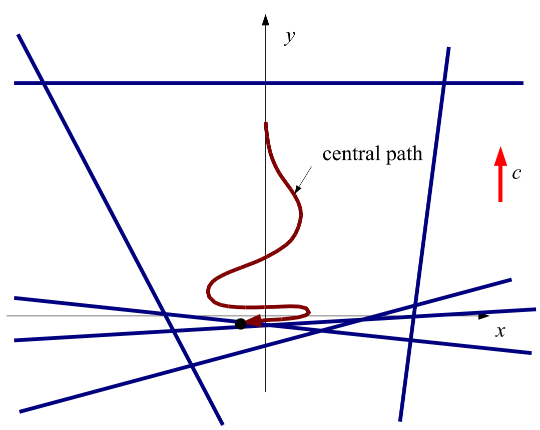

Figure 2: The polytope $P^*_{6,2}$ and its central path

Proposition 3.5. *The total curvature of the central path of* $\min\{y : (x,y) \in P^*_{m,2}\}$ *satisfies*

$$\liminf_{m\to\infty} \frac{\lambda^{(0,1)}(P^*_{m,2})}{m} \geq \pi.$$

Proof. First we show that the central path $\mathcal{P}$ goes through a sequence of $m-2$ points $(x_j, \frac{10^{1-j}}{5})$ for $j = 1, \ldots, m-2$ with $x_j \geq 0$ for odd j and $x_j \leq \frac{-10^{-4}}{m}$ for even j. For $i = 2, \ldots, m$ and $j = 1, \ldots, m-2$, denote z^j_i the first coordinate of the intersection of the line $y = \frac{10^{1-j}}{5}$ and the facet of $P^*_{m,2}$ induced by the i^{th} inequality defining $P^*_{m,2}$, that is, $z_2^j = \frac{10^{-j}}{5} + \frac{1}{2}$, $z_3^j = -\frac{10^{-j+1}}{15} - \frac{1}{3}$, and $z_i^j = (-1)^i(\frac{10^{i-j-1}}{55} + \frac{5}{11} - \frac{10^{-4}}{m}\frac{i}{m})$ for $i = 4, \ldots, m$. As the central path may be characterized as the set of minimizers of the barrier function over appropriate level sets of the objective function, the point $(x_j, \frac{10^{1-j}}{5})$ of $\mathcal{P}$ satisfies $x_j = \arg\max_x \sum_{i=2}^m \ln(-1)^i(z_i^j - x)$. Therefore, to show that $x_j \geq 0$ for odd j and that $x_j \leq \frac{-10^{-4}}{m}$ for even j, it is enough to prove that $g^j(0) > 0$ for odd j and $g^j(\frac{-10^{-4}}{m}) < 0$ for even j where $g^j(x) = \sum_{i=2}^m \frac{d}{dx}\ln(-1)^i(z_i^j - x)$. For simplicity we assume that m is even. A

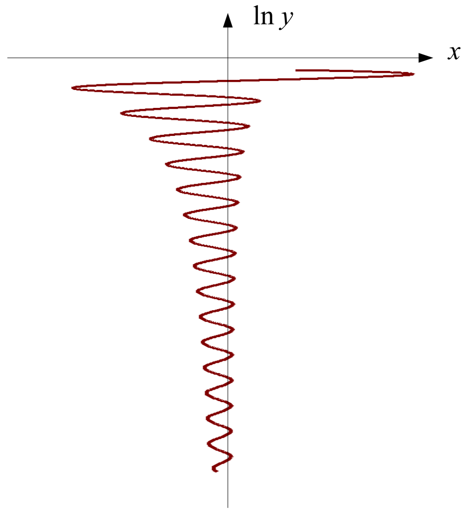

Figure 3: The central path for $P^*_{34,2}$

similar argument applies for odd values of m. Since $(-1)^{k+1}\left(\frac{1}{x-z_k^j} + \frac{1}{x-z_{k+1}^j}\right) > 0$ for $k \geq j+4$ and $\frac{-10^{-4}}{m} \leq x \leq 0$, we have

$$\sum_{i=j+4}^{i=m} \frac{1}{x - z_i^j} \begin{cases} \geq 0, & j \text{ odd}, \quad x = 0, \\ \leq 0, & j \text{ even}, \quad x = \frac{-10^{-4}}{m}. \end{cases} \tag{1}$$

This yields $g^1(0) \geq \frac{-1}{\frac{1}{2}+\frac{1}{50}} + \frac{1}{\frac{1}{3}+\frac{1}{15}} - \frac{1}{\frac{100}{55}+\frac{5}{11}-10^{-4}} = \frac{772}{5639} > 0$. For $j \geq 2$, rewrite

$$g^j(x) = \left(\frac{1}{x - z_2^j} + \frac{1}{x - z_3^j}\right) + \sum_{i=4}^{i<j+2} \frac{1}{x - z_i^j} + \sum_{i=j+2}^{i<j+4} \frac{1}{x - z_i^j} + \sum_{i=j+4}^{i=m} \frac{1}{x - z_i^j}.$$

Observe

$$\frac{1}{x - z_2^j} + \frac{1}{x - z_3^j} = \begin{cases} \frac{-1}{\frac{1}{2}+\frac{10^{-j}}{5}} + \frac{1}{\frac{1}{3}+\frac{10^{-j+1}}{15}} & \text{for } x = 0, \\ \frac{-1}{\frac{1}{2}+\frac{10^{-j}}{5}+\frac{10^{-4}}{m}} + \frac{1}{\frac{1}{3}+\frac{10^{-j+1}}{15}-\frac{10^{-4}}{m}} & \text{for } x = \frac{-10^{-4}}{m}, \end{cases} \tag{2}$$

and

$$\sum_{i=j+2}^{i<j+4} \frac{1}{x - z_i^j} \begin{cases} \geq \frac{1}{\frac{10}{55}+\frac{5}{11}} - \frac{1}{\frac{100}{55}+\frac{5}{11}-\frac{10^{-4}}{m}}, & j \geq 3 \text{ odd}, \quad x = 0, \\ \leq \frac{-1}{\frac{10}{55}+\frac{5}{11}+\frac{10^{-4}}{m}} + \frac{1}{\frac{100}{55}+\frac{5}{11}-2\frac{10^{-4}}{m}}, & j \leq m-4 \text{ even}, \quad x = \frac{-10^{-4}}{m}, \\ \leq \frac{-1}{\frac{10}{55}+\frac{5}{11}+\frac{10^{-4}}{m}}, & j = m-2, \quad x = \frac{-10^{-4}}{m}. \end{cases} \tag{3}$$

For odd $j \geq 3$ and $x = 0$, we have

$$\sum_{i=4}^{i<j+2} \frac{1}{x - z_i^j} \geq -\frac{1}{\frac{10^{3-j}}{55} + \left(\frac{5}{11} - \frac{10^{-4}}{m}\right)} + \frac{1}{\frac{10^{4-j}}{55} + \frac{5}{11}} + \cdots - \frac{1}{\frac{1}{55} + \left(\frac{5}{11} - \frac{10^{-4}}{m}\right)}$$

$$= \frac{-1}{\frac{5}{11} - \frac{10^{-4}}{m}} \left(\frac{1}{1 + \frac{10^{3-j}}{55\left(\frac{5}{11}-\frac{10^{-4}}{m}\right)}} + \frac{1}{1 + \frac{10^{5-j}}{55\left(\frac{5}{11}-\frac{10^{-4}}{m}\right)}} + \cdots + \frac{1}{1 + \frac{1}{55\left(\frac{5}{11}-\frac{10^{-4}}{m}\right)}} \right)$$

$$+\frac{11}{5}\left(\frac{1}{1+\frac{11\cdot 10^{4-j}}{5\cdot 55}} + \frac{1}{1+\frac{11\cdot 10^{6-j}}{5\cdot 55}} + \cdots + \frac{1}{1+\frac{11\cdot 10^{-1}}{5\cdot 55}} \right)$$

$$\geq \frac{-1}{\frac{5}{11} - \frac{10^{-4}}{m}} \left(1 - \frac{10^{3-j}}{55\left(\frac{5}{11}-\frac{10^{-4}}{m}\right)} + \left(\frac{10^{3-j}}{55\left(\frac{5}{11}-\frac{10^{-4}}{m}\right)}\right)^2 \right.$$

$$\left. +1 - \frac{10^{5-j}}{55\left(\frac{5}{11}-\frac{10^{-4}}{m}\right)} + \left(\frac{10^{5-j}}{55\left(\frac{5}{11}-\frac{10^{-4}}{m}\right)}\right)^2 + \cdots + 1 - \frac{1}{55\left(\frac{5}{11}-\frac{10^{-4}}{m}\right)} + \left(\frac{1}{55\left(\frac{5}{11}-\frac{10^{-4}}{m}\right)}\right)^2 \right)$$

$$+\frac{11}{5}\left(1 - \frac{11\cdot 10^{4-j}}{5\cdot 55} + \cdots + 1 - \frac{11\cdot 10^{-1}}{5\cdot 55}\right)$$

$$= \frac{-1}{\frac{5}{11} - \frac{10^{-4}}{m}} \left(\left\lfloor \frac{j}{2} \right\rfloor - \frac{1}{55\left(\frac{5}{11}-\frac{10^{-4}}{m}\right)} \cdot \frac{1 - .01^{\lfloor \frac{j}{2} \rfloor}}{1 - .01} + \left(\frac{1}{55\left(\frac{5}{11}-\frac{10^{-4}}{m}\right)}\right)^2 \frac{1 - .0001^{\lfloor \frac{j}{2} \rfloor}}{1 - .0001} \right)$$

$$+\frac{11}{5}\left(\left\lfloor \frac{j}{2} \right\rfloor - 1 - \frac{11}{550} \cdot \frac{1 - .01^{\lfloor \frac{j}{2} \rfloor - 1}}{1 - .01} \right)$$

$$\geq \frac{-\left\lfloor \frac{j}{2} \right\rfloor \frac{10^{-4}}{m}}{\left(\frac{5}{11}\right)^2 - \frac{5}{11}\frac{10^{-4}}{m}} + \frac{1}{55\left(\frac{5}{11}-\frac{10^{-4}}{m}\right)^2} - \frac{1}{55^2\left(\frac{5}{11}-\frac{10^{-4}}{m}\right)^3 .9999} - \frac{11}{5} - \left(\frac{11}{5}\right)^2 \frac{1}{550\cdot .9999},$$

where the second inequality is based on $1-v \leq \frac{1}{1+v} \leq 1-v+v^2$, $v \geq 0$ and the last equality is obtained by summing up the terms in three resulting geometric series. This, combined with observations (1), (2) and (3), gives, for odd $j \geq 3$,

$$g^j(0) \geq \left(-2 + \frac{1}{\frac{1}{3} + \frac{1}{1500}}\right) + \left(\frac{1}{\frac{10}{55} + \frac{5}{11}} - \frac{1}{\frac{100}{55} + \frac{5}{11} - .0001}\right)$$

$$+ \left(\frac{-.00005}{\left(\frac{5}{11}\right)^2 - \frac{5}{11}\cdot .0001} + \frac{1}{55\left(\frac{5}{11}\right)^2} - \frac{1}{55^2\left(\frac{5}{11} - .0001\right)^3 .9999} - \frac{11}{5} - \left(\frac{11}{5}\right)^2 \frac{1}{550\cdot .9999} \right) = \frac{49}{63838} > 0.$$

Similarly for even $j \geq 2$ and $x = \frac{-10^{-4}}{m}$ we have

$$\sum_{i=4}^{i<j+2} \frac{1}{x - z_i^j} \leq \frac{-1}{\frac{5}{11} + \frac{10^{-4}}{m}} \left(\left\lfloor \frac{j}{2} \right\rfloor - 1 - \frac{1}{550\left(\frac{5}{11} + \frac{10^{-4}}{m}\right)} \cdot \frac{1 - .01^{\lfloor \frac{j}{2} \rfloor - 1}}{1 - .01} \right)$$

$$+ \frac{1}{\frac{5}{11} - 2\frac{10^{-4}}{m}} \left(\left\lfloor \frac{j}{2} \right\rfloor - 1 - \frac{1}{55\left(\frac{5}{11} - 2\frac{10^{-4}}{m}\right)} \cdot \frac{1 - .01^{\lfloor \frac{j}{2} \rfloor - 1}}{1 - .01} + \left(\frac{1}{55\left(\frac{5}{11} - 2\frac{10^{-4}}{m}\right)} \right)^2 \frac{1 - .0001^{\lfloor \frac{j}{2} \rfloor - 1}}{1 - .0001} \right)$$

$$\leq \left(\left\lfloor \frac{j}{2} \right\rfloor - 1 \right) \frac{2\frac{10^{-4}}{m} + \frac{10^{-4}}{m}}{\left(\frac{5}{11}\right)^2 - \left(\frac{-10^{-4}}{m}\right)^2 - \frac{10^{-4}}{m}\left(\frac{5}{11} + \frac{10^{-4}}{m}\right)} + \frac{1}{550\left(\frac{5}{11} + \frac{10^{-4}}{m}\right)^2} \cdot \frac{1}{1 - .01}$$

$$- \frac{1}{55\left(\frac{5}{11} - 2\frac{10^{-4}}{m}\right)^2} + \frac{1}{55^2\left(\frac{5}{11} - 2\frac{10^{-4}}{m}\right)^3} \cdot \frac{1}{1 - .0001}.$$

Thus, for even $j \geq 2$.

$$g^j\left(\frac{-10^{-4}}{m}\right) \leq \left(\frac{-1}{\frac{1}{2} + \frac{1}{500} + .0001} + \frac{1}{\frac{1}{3} - .0001} \right) + \left(\frac{-1}{\frac{10}{55} + \frac{5}{11} + .0001} + \frac{1}{\frac{100}{55} + \frac{5}{11} - .0002} \right)$$

$$+ \left(\frac{.00015}{\left(\frac{5}{11}\right)^2 - .0001^2 - .0001\left(\frac{5}{11} + .0001\right)} - \frac{89}{99} \cdot \frac{1}{55\left(\frac{5}{11}\right)^2} + \frac{1}{55^2 \cdot .999} \cdot \frac{1}{\left(\frac{5}{11} - .0002\right)^3} \right) = \frac{-784}{3985} < 0.$$

Therefore, the central path $\mathcal{P}$ goes through a sequence of $m-2$ points (x_j, y_j) with $y_j = \frac{10^{1-j}}{5}$ and $x_j \geq 0$ for odd j, $x_j \leq \frac{-10^{-4}}{m}$ for even j. One can easily check that $(x_j, y_j) \in \mathcal{P}$ for $j = 1, \ldots, m-2$ by verifying that the analytic center χ is above the line $y = \frac{1}{5}$. We have

$$\chi = (\chi_1, \chi_2) = \arg \max_{(x,y) \in P^*_{m,2}} \left(\ln(1-y) + \ln\left(-x + \frac{y}{10} + \frac{1}{2}\right) + \ln\left(x + \frac{y}{3} + \frac{1}{3}\right) \right.$$
$$\left. + \sum_{i=4}^{m} \ln\left((-1)^{i+1}x + \frac{10^{i-2}y}{11} + \frac{5}{11} - \frac{10^{-4}}{m}\frac{i}{m} \right) \right).$$

Therefore, to show that $\chi_2 > \frac{1}{5}$, it is enough to prove that the derivative with respect to y of the log-barrier function is negative for $(x, y) \in P^*_{m,2}$ and $y \leq \frac{1}{5}$, that is,

$$\frac{-1}{1-y} + \frac{1}{-10x + y + 5} + \frac{1}{3x + y + 1} + \sum_{i=4}^{m} \frac{10^{i-2}}{\left((-1)^{i+1}11x + 10^{i-2}y + 5 - 11 \cdot \frac{10^{-4}}{m}\frac{i}{m}\right)} > 0,$$

which is implied by $\frac{-1}{1-y} + \frac{100}{-11x + 100y + 5 - 11 \cdot \frac{.0001}{m}\frac{4}{m}} > \frac{-5}{4} + \frac{100}{\frac{100}{5} + 5 + \frac{66}{15}} = \frac{1265}{588} > 0.$

To show that $\liminf_{m\to\infty} \frac{\lambda^{(0,1)^T}(P^*_{m,2})}{m} \geq \pi$, consider three consecutive points from this sequence, say $(x_{j-1}, y_{j-1}), (x_j, y_j), (x_{j+1}, y_{j+1})$, and observe that for any $\varepsilon > 0$ we can choose m so that for all $\varepsilon m \leq j < m-2$ we have $\frac{|y_j - y_{j-1}|}{|x_j - x_{j-1}|} < \varepsilon$, $\frac{|y_{j+1} - y_j|}{|x_{j+1} - x_j|} < \varepsilon$. Let m be such a value and $j \geq \varepsilon m$. Without loss off generality j might be assumed odd and let $\tau_{j-1}, \tau_j, \tau_{j+1} \in \mathbb{R}$ be such that $\mathcal{P}_{\text{arc}}(\tau_k) = (x_k, y_k)$, $k = j-1, j, j+1$. We show by contradiction that there is a t_1 such that the first coordinate $\left(\dot{\mathcal{P}}_{\text{arc}}(t_1)\right)_1 > \sqrt{1-\varepsilon^2}$. Suppose that for all $t \in [\tau_{j-1}, \tau_j]$ we have $\left(\dot{\mathcal{P}}_{\text{arc}}(t)\right)_1 \leq \sqrt{1-\varepsilon^2}$, then

$\left(\dot{\mathcal{P}}_{\rm arc}(t)\right)_2 \le -\varepsilon$ since $\|\dot{\mathcal{P}}_{\rm arc}(t)\| = 1$ and $(\mathcal{P}_{\rm arc}(t))_2$ is monotone-decreasing with respect to t. By the Mean-Value Theorem it follows that $\tau_j - \tau_{j-1} > x_j - x_{j-1}$, and thus, by the same theorem, we must have $(\mathcal{P}_{\rm arc}(\tau_j))_2 - (\mathcal{P}_{\rm arc}(\tau_{j-1}))_2 = y_j - y_{j-1} < -\varepsilon(x_j - x_{j-1})$, a contradiction. Similarly, there is a t_2 such that $\left(\dot{\mathcal{P}}_{\rm arc}(t_2)\right)_1 < -\sqrt{1-\varepsilon^2}$. Since the total curvature K_j of the segment of $\mathcal{P}_{\rm arc}$ connecting the points $(x_{j-1}, y_{j-1}), (x_j, y_j), (x_{j+1}, y_{j+1})$ corresponds to the length of the curve $\dot{\mathcal{P}}_{\rm arc}$ connecting the corresponding derivative points on a unit 2-sphere, K_j may be bounded below by the length of the geodesic between the points $\dot{\mathcal{P}}_{\rm arc}(t_1)$ and $\dot{\mathcal{P}}_{\rm arc}(t_2)$, that is, bounded below by a constant arbitrarily close to π. Now simply add all K_j for all $\varepsilon m \le j < m-2$. □

Holt and Klee [9] showed that, for $m > n \ge 13$, the conjecture of Hirsch is tight. Fritzsche and Holt [8] extended the result to $m > n \ge 8$. Since the polytope $P^*_{m,2}$ can be generalized to higher dimensions by adding the box constraints $0 \le x_i \le 1$ for $i \ge 3$, we have:

Corollary 3.1. (Continuous analogue of the result of Holt and Klee) $\liminf_{m\to\infty} \frac{\Lambda(m,n)}{m} \ge \pi$, *that is,* $\Lambda(m,n)$ *is bounded below by a constant times* m.

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