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Title:

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Abstract

The Klee-Minty cube is a well-known worst case example for which the simplex method takes an exponential number of iterations as the algorithm visits all the 2^n vertices of the n-dimensional cube. While such behavior is excluded by polynomial interior point methods, we show that, by adding an exponential number of redundant inequalities, the central path can be bent along the edges of the Klee-Minty cube. More precisely, for an arbitrarily small δ, the central path takes $2^n - 2$ turns as it passes through the δ-neighborhood of all the vertices of the Klee-Minty cube in the same order as the simplex method does.

Key words: Linear programming, central path, Klee-Minty cube.
MSC2000 Subject Classification: Primary: 90C05; Secondary: 90C51, 90C27, 52B12

1 Introduction

While the *simplex method*, introduced by Dantzig [1], works very well in practice for linear optimization problems, Klee and Minty [5] gave an example in 1972 for which the simplex method takes an exponential number of iterations. More precisely, they considered a maximization problem over an n-dimensional squashed cube and proved that a variant of the simplex method visits all its 2^n vertices; that is, the time complexity is not polynomial for the worst case, as $2^n - 1$ iterations are necessary for this n-dimensional linear optimization problem. The pivot rule used in the Klee-Minty example was the most negative reduced cost but variants of the Klee-Minty n-cube showing an exponential running time exist for most pivot rules; see [9] and the references therein. The Klee-Minty worst-case example partially stimulated the search for a polynomial algorithm and, in 1979, Khachiyan's [4] *ellipsoid method* proved that linear programming is indeed polynomially solvable. In 1984, Karmarkar [3] proposed a more efficient polynomial algorithm that sparked the research on polynomial *interior point methods*. In short, while the simplex method goes along the edges of the polyhedron corresponding to the feasible region, interior point methods pass through the interior of this polyhedron. Starting at the *analytic center*, most interior point methods follow the so-called *central path* and converge to the analytic center of the optimal face; see e.g., [6, 7, 8, 11, 12].

In this paper, following the Klee-Minty approach, we show that, by carefully adding an exponential number of redundant constraints to the Klee-Minty n-cube, the central path can be bent along its edges. In other words, we give an example where, for an arbitrarily small δ, starting from the δ-neighborhood of a vertex adjacent to the optimal solution, the central path takes $2^n - 2$ turns as, before converging to the optimal solution, it passes through the δ-neighborhood of all the vertices of the Klee-Minty cube in the same order as the simplex method does.

Before stating the main result in Section 2 and giving its proof in Section 3, we illustrate the bending of the central path in the 2 and 3 dimensional cases. Fig. 1 and Fig. 2 show the trajectory of the central path starting from the highest vertex and converging to the origin after visiting each vertex of the Klee-Minty cube. The redundant constraints correspond to hyperplanes parallel to the facets of the cube containing the origin. More precisely, in dimension 2, the redundant inequality $16 + x_1 \geq 0$ is added 15 360 times and the redundant inequality $16 + x_2 - \frac{1}{4}x_1 \geq 0$ is added 40 960 times. Starting from the highest vertex and with $\delta = 0.1$, the central path visits the δ-neighborhood of each vertex of the Klee-Minty cube in the same order as the simplex algorithm does before converging to the optimal solution; that is, the origin. In dimension 3, the redundant inequality $48 + x_1 \geq 0$ (resp. $48 + x_2 - \frac{1}{4}x_1 \geq 0$ and $48 + x_3 - \frac{1}{4}x_2 \geq 0$) is added 161 280 (resp. 552 960, and 1 474 560 times).

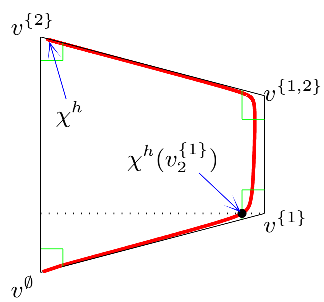

Figure 1: Central path nearing all the vertices of the Klee-Minty 2-cube.

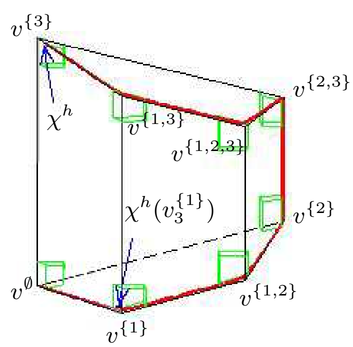

Figure 2: Central path nearing all the vertices of the Klee-Minty 3-cube.

2 Notations and the Main Result

We consider the following Klee-Minty variant where ε is a small positive factor by which the unit cube $[0,1]^n$ is squashed.

$$\begin{array}{rl} \min & x_n \\ \text{subject to} & 0 \le x_1 \le 1 \\ & \varepsilon\, x_{k-1} \le x_k \le 1 - \varepsilon\, x_{k-1} \quad \text{for } k = 2, \ldots, n. \end{array}$$

We denote this linear optimization problem by KM. This minimization problem has $2n$ constraints, n variables and the feasible region is an n-dimensional cube denoted by $\mathcal{C}$. Some variants of the simplex method take $2^n - 1$ iterations to solve KM as they visit all the vertices ordered by the decreasing value of the last coordinate x_n starting from $v^{\{n\}} = (0, \ldots, 0, 1)$ till the optimal value $x_n^* = 0$ is reached at the origin $v^{\emptyset}$. If an interior point method is used to solve KM, the central path starts from the analytic center χ of $\mathcal{C}$ and converges to the origin as it is shown in Fig. 3.

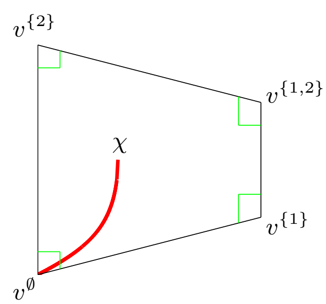

Figure 3: The central path in the non-redundant Klee-Minty 2-cube.

While adding a set h of redundant inequalities does not change the feasible region of KM, the analytic center χ^h and the central path are affected by the addition of redundant constraints. We consider redundant inequalities induced by hyperplanes parallel to the n facets of $\mathcal{C}$ containing the origin. To ease the analysis, we consider that all redundant hyperplanes are put at the same distance d to the corresponding parallel facet of $\mathcal{C}$. The constraint parallel to $H_1 : x_1 = 0$ is added h_1 times and the constraint parallel to $H_k : x_k = \varepsilon x_{k-1}$ is added h_k times for $k = 2, \ldots, n$. By abuse of notation, the set h is denoted by the integer-vector $h = (h_1, \ldots, h_n)$. With these notations, the redundant linear optimization problem KM^h is defined by

$$\begin{array}{rrl}
\min & x_n & \\
\text{subject to} & 0 \le x_1 \le 1 & \\
& \varepsilon\, x_{k-1} \le x_k \le 1 - \varepsilon\, x_{k-1} & \text{for } k = 2, \dots, n \\
& 0 \le d + x_1 & \text{repeated } h_1 \text{ times} \\
& \varepsilon\, x_1 \le d + x_2 & \text{repeated } h_2 \text{ times} \\
& \vdots & \vdots \\
& \varepsilon\, x_{n-1} \le d + x_n & \text{repeated } h_n \text{ times.}
\end{array}$$

To give a flavor of the main result, we first present Lemma 2.1 stating that, by adding $\frac{d+1}{\varepsilon^{n-1}\delta}$ times the redundant inequality $\varepsilon x_{n-1} \le d + x_n$ to the original KM formulation, the analytic center χ^h can be pushed arbitrarily close to the vertex $v^{\{n\}} = (0, \dots, 0, 1)$. To warranty, without loss of generality, that h is integer valued, we assume that both ε^{-1} and δ^{-1} are positive integers.

Lemma 2.1. *Given $\delta \le \varepsilon \le \frac{1}{4}$, d positive integer and $h = (0, \dots, 0, \frac{d+1}{\varepsilon^{n-1}\delta})$, the analytic center χ^h satisfies $|\chi^h - v^{\{n\}}|_\infty \le \delta$.*

While Lemma 2.1 sets the starting point of the central path in the δ-neighborhood of $v^{\{n\}}$, Proposition 2.2 states that, for a careful choice of d and h, the central path of the cube takes $2^n - 2$ turns before converging to the origin as it passes through the δ-neighborhood of all the 2^n vertices of the Klee-Minty n-cube.

Proposition 2.2. *Given $\varepsilon \le \frac{1}{4}$, $\delta < \varepsilon^{n-1}$, choose integer $d \ge n2^{n+1}$ and $h = \frac{4nd}{\delta}(\frac{2^n-1}{\varepsilon}, \dots, \frac{2^n-2^{k-1}}{\varepsilon^k}, \dots, \frac{2^{n-1}}{\varepsilon^n})$, then for each vertex v^S of the Klee-Minty n-cube, there is a point $\chi^h(v_n^S)$ of the central path satisfying $|\chi^h(v_n^S) - v^S|_\infty \le \delta$.*

3 Proofs of Lemma 2.1 and Proposition 2.2

3.1 Proof of Lemma 2.1

The analytic center $\chi^h = (\xi_1^h, \dots, \xi_n^h)$ of KM^h is the solution to the problem consisting of maximizing the product of the slack variables

$$\begin{array}{ll}
s_1 = x_1 & \\
s_k = x_k - \varepsilon x_{k-1} & \text{for } k = 2, \dots, n \\
\bar{s}_1 = 1 - x_1 & \\
\bar{s}_k = 1 - \varepsilon x_{k-1} - x_k & \text{for } k = 2, \dots, n \\
\tilde{s}_1 = d + s_1 & \text{repeated } h_1 \text{ times} \\
\vdots & \vdots \\
\tilde{s}_n = d + s_n & \text{repeated } h_n \text{ times.}
\end{array}$$

Equivalently, χ^h is the solution of the following maximization problem

$$\max_x \sum_{k=1}^{n} (\log s_k + \log \bar{s}_k + h_k \log \tilde{s}_k),$$

i.e., with the convention $x_0 = 0$,

$$\max_x \sum_{k=1}^{n} \Big(\log(x_k - \varepsilon x_{k-1}) + \log(1 - \varepsilon x_{k-1} - x_k) + h_k \log(d + x_k - \varepsilon x_{k-1}) \Big).$$

The optimality conditions (the gradient is equal to zero at optimality) for this concave maximization problem give

$$\left\{ \begin{array}{rl} \frac{1}{\sigma_k^h} - \frac{\varepsilon}{\sigma_{k+1}^h} - \frac{1}{\bar{\sigma}_k^h} - \frac{\varepsilon}{\bar{\sigma}_{k+1}^h} + \frac{h_k}{\tilde{\sigma}_k^h} - \frac{h_{k+1}\varepsilon}{\tilde{\sigma}_{k+1}^h} = 0 & \text{for } k = 1, \dots, n-1 \\ \frac{1}{\sigma_n^h} - \frac{1}{\bar{\sigma}_n^h} + \frac{h_n}{\tilde{\sigma}_n^h} = 0 & \\ \sigma_k^h > 0, \bar{\sigma}_k^h > 0, \tilde{\sigma}_k^h > 0 & \text{for } k = 1, \dots, n, \end{array} \right. \tag{1}$$

where

$$\begin{array}{ll} \sigma_1^h = \xi_1^h & \\ \sigma_k^h = \xi_k^h - \varepsilon \xi_{k-1}^h & \text{for } k = 2, \dots, n \\ \bar{\sigma}_1^h = 1 - \xi_1^h & \\ \bar{\sigma}_k^h = 1 - \varepsilon \xi_{k-1}^h - \xi_k^h & \text{for } k = 2, \dots, n \\ \tilde{\sigma}_k^h = d + \sigma_k^h & \text{for } k = 1, \dots, n. \end{array}$$

The following lemma states that, for h_n large enough relatively to the other h_k values, the analytic center χ^h is pushed to the neighborhood of the vertex $v^{\{n\}} = (0, \dots, 0, 1)$.

Lemma 3.1. *Given $\delta \le \varepsilon \le \frac{1}{4}$ and $h_1, \dots, h_{n-1}$, we have*

$$|\chi^h - v^{\{n\}}|_\infty \le \delta \quad \textit{for} \quad h_n \ge \frac{d+1}{\eta_n} \quad \textit{where} \quad \frac{1}{\eta_n} = \max_{1 \le k \le n-1} \left\{ \frac{1}{2\varepsilon^{n-k}} \left(\frac{h_k}{d} + \frac{2}{\delta} \right) \right\}.$$

Proof. The analytic center $\chi^h = (\xi_1^h, \dots, \xi_n^h)$ is the solution of (1). Let us consider the n-th equation of (1). Since $\tilde{\sigma}_n^h \le d + 1$, we have: $h_n \le \frac{d+1}{\bar{\sigma}_n^h}$. Thus, as $h_n \ge \frac{d+1}{\eta_n}$, we have $\bar{\sigma}_n^h \le \eta_n$, which implies $\bar{\sigma}_n^h \le \frac{\delta}{2}$. Let us then consider the $(n-1)$-th equation. We have

$$\frac{1}{\sigma_{n-1}^h} = \frac{1}{\bar{\sigma}_{n-1}^h} + \frac{\varepsilon}{\sigma_n^h} + \frac{\varepsilon}{\bar{\sigma}_n^h} - \frac{h_{n-1}}{\tilde{\sigma}_{n-1}^h} + \frac{h_n \varepsilon}{\tilde{\sigma}_n^h} \ge \frac{\varepsilon}{\bar{\sigma}_n^h} - \frac{h_{n-1}}{\tilde{\sigma}_{n-1}^h} + \frac{h_n \varepsilon}{\tilde{\sigma}_n^h}.$$

Since $\bar{\sigma}_n^h \le \eta_n$, $\tilde{\sigma}_{n-1}^h \ge d$ and $\tilde{\sigma}_n^h \le d + 1$, this implies

$$\frac{1}{\sigma_{n-1}^h} \ge \frac{2\varepsilon}{\eta_n} - \frac{h_{n-1}}{d} \ge \frac{2}{\delta},$$

i.e., $\sigma^h_{n-1} \leq \frac{\delta}{2}$. The first $n-1$ equations of (1) can be rewritten as

$$\frac{1}{\sigma^h_k} + \frac{h_k}{\tilde{\sigma}^h_k} = \frac{1}{\bar{\sigma}^h_k} + \varepsilon\left(\frac{1}{\sigma^h_{k+1}} + \frac{h_{k+1}}{\tilde{\sigma}^h_{k+1}}\right) + \frac{\varepsilon}{\bar{\sigma}^h_{k+1}} \qquad \text{for } k = 1, \ldots, n-1.$$

For $k \leq n-2$, forward substitutions for the k-th, $(k+1)$-th,... $(n-1)$-th equations give

$$\frac{1}{\sigma^h_k} + \frac{h_k}{\tilde{\sigma}^h_k} = \frac{1}{\bar{\sigma}^h_k} + 2\sum_{j=k+1}^{n-1} \frac{\varepsilon^{j-k}}{\bar{\sigma}^h_j} + \frac{\varepsilon^{n-k}}{\sigma^h_n} + \frac{h_n\varepsilon^{n-k}}{\tilde{\sigma}^h_n} + \frac{\varepsilon^{n-k}}{\bar{\sigma}^h_n},$$

which implies

$$\frac{1}{\sigma^h_k} \geq \frac{\varepsilon^{n-k}}{\bar{\sigma}^h_n} - \frac{h_k}{\tilde{\sigma}^h_k} + \frac{h_n\varepsilon^{n-k}}{\tilde{\sigma}^h_n}.$$

Since $\bar{\sigma}^h_n \leq \eta_n$, $\tilde{\sigma}^h_k \geq d$ and $\tilde{\sigma}^h_n \leq d+1$, this implies

$$\frac{1}{\sigma^h_k} \geq \frac{2\varepsilon^{n-k}}{\eta_n} - \frac{h_k}{d} \geq \frac{2}{\delta},$$

i.e., $\sigma^h_k \leq \frac{\delta}{2}$ for $k = 1, \ldots, n-2$. Therefore, for $h_n \geq \frac{d+1}{\eta_n}$, we have: $\xi^h_k \leq \delta$ for $k = 1, \ldots, n-1$ and $1 - \xi^h_n \leq \delta$. □

Lemma 2.1 is a direct corollary of Lemma 3.1 with $h = (0, \ldots, 0, h_n)$. □

3.2 Proof of Proposition 2.2

3.2.1 Preliminary Lemmas

Lemma 3.2. *Given $\delta \leq \varepsilon \leq \frac{1}{4}$ and integer $d \geq n2^{n+1}$, then $h = \frac{4nd}{\delta}(\frac{2^n-1}{\varepsilon}, \ldots, \frac{2^n-2^{k-1}}{\varepsilon^k}, \ldots, \frac{2^{n-1}}{\varepsilon^n})$ is a positive and integer solution of $Ah \geq b$, where $b = \frac{4n}{\delta}(1, \ldots, 1)$ and*

$$A = \begin{pmatrix} \frac{-1}{d} & 0 & 0 & 0 & \cdots & 0 & \frac{2\varepsilon^{n-1}}{d+1} \\ \frac{1}{d+1} & \frac{-\varepsilon}{d} & 0 & 0 & \cdots & 0 & 0 \\ \frac{-1}{d} & \frac{2\varepsilon}{d+1} & \frac{-\varepsilon^2}{d} & 0 & \cdots & 0 & 0 \\ \frac{-1}{d} & \frac{-\varepsilon}{d(d+1)} & \frac{2\varepsilon^2}{d+1} & \frac{-\varepsilon^3}{d} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{-1}{d} & \frac{-\varepsilon}{d(d+1)} & \frac{-\varepsilon^2}{d(d+1)} & \frac{-\varepsilon^3}{d(d+1)} & \cdots & \frac{-\varepsilon^{n-2}}{d} & 0 \\ \frac{-1}{d} & \frac{-\varepsilon}{d(d+1)} & \frac{-\varepsilon^2}{d(d+1)} & \frac{-\varepsilon^3}{d(d+1)} & \cdots & \frac{2\varepsilon^{n-2}}{d+1} & \frac{-\varepsilon^{n-1}}{d} \end{pmatrix}.$$

Proof. Multiplying both sides of $Ah \geq b$ by $\frac{\delta\varepsilon(d+1)}{4n}$, we have

$$\begin{cases} d(1-\varepsilon) & \geq 2^n - 1 + \varepsilon \\ d(1-\varepsilon) & \geq 2^n - 2 + \varepsilon \\ d(1-\varepsilon) & \geq 2^{n+1} - 5 + \varepsilon \\ d(1-\varepsilon) & \geq 2^n - 2^k + \sum_{j=1}^{k-1}\left(2^n - 2^{j-1}\right) + \varepsilon \qquad \text{for } k = 4, \ldots, n, \end{cases}$$

which, since $d(1-\varepsilon) \geq n2^n$, is implied by the obvious conditions

$$\begin{cases} n2^n & \geq 2^n - 1 + \varepsilon \\ n2^n & \geq 2^n - 2 + \varepsilon \\ n2^n & \geq 2^{n+1} - 5 + \varepsilon \\ n2^n & \geq k2^n - 3 \cdot 2^{k-1} + 1 + \varepsilon \qquad \text{for } k = 4, \dots, n. \end{cases}$$

□

To ease the notations, we define, for, $k = 1, \dots, n-1$

$$\begin{aligned} \ell_k = & \ \tfrac{h_k}{d+1} - \tfrac{h_{k+1}\varepsilon}{d} \\ u_k = & \ \tfrac{h_k}{d} - \tfrac{h_{k+1}\varepsilon}{d+1}. \end{aligned}$$

Lemma 3.3. *The system $Ah \geq b$ is equivalent to*

$$\begin{cases} \tfrac{2h_n\varepsilon^{n-1}}{d+1} - \tfrac{h_1}{d} \geq \tfrac{4n}{\delta} \\ \ell_1 \geq \tfrac{4n}{\delta} \\ \ell_k\varepsilon^{k-1} - \sum_{j=1}^{k-1} u_j\varepsilon^{j-1} \geq \tfrac{4n}{\delta} \qquad \textit{for } k = 2, \dots, n-1. \end{cases}$$

Proof. The first 2 inequalities are direct reformulations of the first 2 inequalities of the system $Ah \geq b$. For $k = 2, \dots, n-1$, the inequality: $\ell_k\varepsilon^{k-1} - \sum_{j=1}^{k-1} u_j\varepsilon^{j-1} \geq \frac{4n}{\delta}$ can be rewritten as: $\frac{2h_k\varepsilon^{k-1}}{d+1} - \frac{h_{k+1}\varepsilon^k}{d} - \frac{h_1}{d} - \frac{1}{d(d+1)}\sum_{j=2}^{k-1} h_j\varepsilon^{j-1} \geq \frac{4n}{\delta}$. □

Corollary 3.4. *For h satisfying the last $n-1$ inequalities of $Ah \geq b$, we have*

$$\ell_k\varepsilon^{k-1} \geq \frac{2^{k+1}n}{\delta} \qquad \textit{for } k = 1, \dots, n-1.$$

Proof. The proof is by induction on k. We have $\ell_1 \geq \frac{4n}{\delta}$ and the result follows by using $u_k \geq \ell_k$ in Lemma 3.3. □

Corollary 3.5. *Given $\delta \leq \varepsilon \leq \frac{1}{4}$ and a positive integer h satisfying the last $n-1$ inequalities of $Ah \geq b$, we have*

$$|\chi^h - v^{\{n\}}|_\infty \leq \delta \quad \textit{for} \quad \frac{2h_n\varepsilon^{n-1}}{d+1} \geq \frac{h_1}{d} + \frac{2}{\delta}.$$

Proof. Corollary 3.4 implies $\ell_k \geq 0$ for $k = 1, \dots, n-1$. Hence, in Lemma 3.1 we have $\frac{1}{\eta_n} = \frac{1}{2\varepsilon^{n-1}}\left(\frac{h_1}{d} + \frac{2}{\delta}\right)$ which gives the result. □

The central path of KM^h can be defined as the set of analytic centers $\chi^h(\alpha) = (x_1^h, \dots, x_{n-1}^h, \alpha)$ of the intersection of the hyperplane $H_\alpha : x_n = \alpha$ with the feasible region of KM^h where $0 \leq \alpha \leq \xi_n^h$, see [7]. These intersections are called the *α-level sets* and $\chi^h(\alpha)$ is the solution of the following system

$$\begin{cases} \frac{1}{s_k^h} - \frac{\varepsilon}{s_{k+1}^h} - \frac{1}{\bar{s}_k^h} - \frac{\varepsilon}{\bar{s}_{k+1}^h} + \frac{h_k}{\tilde{s}_k^h} - \frac{h_{k+1}\varepsilon}{\tilde{s}_{k+1}^h} = 0 \qquad \text{for } k = 1, \dots, n-1 \\ s_k^h > 0, \ \bar{s}_k^h > 0, \tilde{s}_k^h > 0 \qquad \text{for } k = 1, \dots, n-1, \end{cases} \tag{2}$$

where

$$\begin{array}{rcll} s_1^h &=& x_1^h & \\ s_k^h &=& x_k^h - \varepsilon x_{k-1}^h & \text{for } k = 2,\dots,n-1 \\ s_n^h &=& \alpha - \varepsilon x_{n-1} & \\ \bar{s}_1^h &=& 1 - x_1^h & \\ \bar{s}_k^h &=& 1 - \varepsilon x_{k-1}^h - x_k^h & \text{for } k = 2,\dots,n-1 \\ \bar{s}_n^h &=& 1 - \alpha - \varepsilon x_{n-1}^h & \\ \tilde{s}_k^h &=& d + s_k^h & \text{for } k = 1,\dots,n. \end{array}$$

In the rest of the paper, we assume that $\delta \le \varepsilon^{n-1}$ and that a positive integer h satisfying $Ah \ge b$ is given. Corollary 3.5 implies that $1 - \delta \le \xi_n^h$ and therefore we can consider the α-level set for $\alpha \le 1 - \varepsilon^{n-1}$ as it implies $\alpha \le \xi_n^h$.

Lemma 3.6. *Given $\varepsilon \le \frac{1}{4}$, $\delta < \varepsilon^{n-1}$, integer $d \ge n2^{n+1}$ and a positive integer h satisfying $Ah \ge b$; for $0 \le \alpha \le 1 - \varepsilon^{n-1}$ and $k \notin \{1, n\}$, if the k-th coordinate x_k^h of the analytic center $\chi^h(\alpha)$ satisfies*

$$x_k^h \in \left[\varepsilon^{k-1} - t_k\varepsilon^{k-1}\delta,\ 1 - \varepsilon^{k-1} + t_k\varepsilon^{k-1}\delta\right], \text{ where } \begin{cases} t_1 = 1 \\ t_2 = 1 - \frac{2}{4n-1} \\ t_{k+1} = t_k - \frac{1}{n} \quad \text{for } k = 2,\dots,n-1, \end{cases}$$

then, $x_{\hat{k}}^h \ge 1 - \varepsilon$ for some $\hat{k}$ smaller than or equal to $k-1$.

Proof. Assume to the contrary that the statement is false, i.e., $x_{\hat{k}}^h < 1-\varepsilon$ for $\hat{k} = 1,\dots,k-1$. Considering the first equation of (2) and successively using $x_1^h < 1-\varepsilon$, $\delta < \varepsilon$ and Lemma 3.3, we have

$$\frac{\varepsilon}{s_2^h} + \frac{\varepsilon}{\bar{s}_2^h} = \frac{1}{s_1^h} - \frac{1}{\bar{s}_1^h} + \frac{h_1}{\tilde{s}_1^h} - \frac{h_2\varepsilon}{\tilde{s}_2^h} \ge -\frac{1}{\varepsilon} + \frac{h_1}{d+1} - \frac{h_2\varepsilon}{d} \ge -\frac{1}{\delta} + \frac{h_1}{d+1} - \frac{h_2\varepsilon}{d} \ge \frac{4n-1}{\delta},$$

which implies either

$$x_2^h \le \varepsilon x_1^h + \frac{2\varepsilon\delta}{4n-1} < \varepsilon(1-\varepsilon) + \frac{2\varepsilon\delta}{4n-1} \le \varepsilon(1-\delta) + \frac{2\varepsilon\delta}{4n-1} = \varepsilon - t_2\varepsilon\delta,$$

or

$$x_2^h \ge 1 - \varepsilon x_1^h - \frac{2\varepsilon\delta}{4n-1} > 1 - \varepsilon(1-\varepsilon) - \frac{2\varepsilon\delta}{4n-1} > 1 - \varepsilon.$$

Since $x_2^h < 1-\varepsilon$, this implies $x_2^h < \varepsilon - t_2\varepsilon\delta$. Similarly, considering the $\hat{k}$-th equation of (2) for $\hat{k} = 2,\dots,k-2$, we have: $x_{\hat{k}}^h < \varepsilon^{\hat{k}-1} - t_{\hat{k}}\varepsilon^{\hat{k}-1}\delta$. Considering the $(k-1)$-st equation and successively using $x_{k-1}^h < \varepsilon^{k-1}$, $x_{k-2}^h < 1$ and Corollary 3.4, we have

$$\frac{\varepsilon}{s_k^h} + \frac{\varepsilon}{\bar{s}_k^h} = \frac{1}{s_{k-1}^h} - \frac{1}{\bar{s}_{k-1}^h} + \frac{h_{k-1}}{\tilde{s}_{k-1}^h} - \frac{h_k\varepsilon}{\tilde{s}_k^h} \ge \frac{1}{\varepsilon^{k-2}} - \frac{1}{1-\varepsilon^{k-2}-\varepsilon} + \frac{h_{k-1}}{d+1} - \frac{h_k\varepsilon}{d} \ge \ell_{k-1} \ge \frac{2^k n}{\varepsilon^{k-2}\delta},$$

which implies either

$$x_k^h \le \varepsilon x_{k-1}^h + \frac{\varepsilon^{k-1}\delta}{2^{k-1}n} < \varepsilon(\varepsilon^{k-2} - t_{k-1}\varepsilon^{k-2}\delta) + \frac{\varepsilon^{k-1}\delta}{2^{k-1}n} \le \varepsilon^{k-1} - t_k\varepsilon^{k-1}\delta,$$

or

$$x_k^h \ge 1 - \varepsilon x_{k-1}^h - \frac{\varepsilon^{k-1}\delta}{2^{k-1}n} > 1 - \varepsilon(\varepsilon^{k-2} - t_{k-1}\varepsilon^{k-2}\delta) - \frac{\varepsilon^{k-1}\delta}{2^{k-1}n} \ge 1 - \varepsilon^{k-1} + t_k\varepsilon^{k-1}\delta.$$

This is impossible as $x_k^h \in \left[\varepsilon^{k-1} - t_k\varepsilon^{k-1}\delta,\ 1 - \varepsilon^{k-1} + t_k\varepsilon^{k-1}\delta\right]$. □

Lemma 3.7. *Given $\varepsilon \le \frac{1}{4}$, $\delta < \varepsilon^{n-1}$, $d \ge n2^{n+1}$, a positive integer h satisfying $Ah \ge b$ and $t_1, \dots, t_n$ as specified in Lemma 3.6; for $0 \le \alpha \le 1 - \varepsilon^{n-1}$ and $k \notin \{1, n\}$, if the k-th coordinate x_k^h of the analytic center $\chi^h(\alpha)$ satisfies*

$$x_k^h \in \left[t_k\varepsilon^{k-1}\delta,\ 1 - t_k\varepsilon^{k-1}\delta\right],$$

then $s_{\hat{k}}^h \ge \frac{1}{2^k n}\varepsilon^{\hat{k}-1}\delta$ for some $\hat{k}$ smaller than or equal to $k-1$.

Proof. Assume to the contrary that the statement is false, i.e., $s_{\hat{k}}^h < \frac{1}{2^k n}\varepsilon^{\hat{k}-1}\delta$ for $\hat{k} = 1, \dots, k-1$. This implies

$$x_{\hat{k}}^h < \frac{\varepsilon^{\hat{k}-1}\delta}{4n}\sum_{j=1}^{\hat{k}-1}\frac{1}{2^{j-2}} \qquad \text{for } \hat{k} = 1, \dots, k-1.$$

Considering the $(k-1)$-st equation of (2) and using $\bar{s}_{k-1}^h = 1 - 2\varepsilon x_{k-2}^h - s_{k-1}^h$, we have

$$\frac{\varepsilon}{s_k^h} + \frac{\varepsilon}{\bar{s}_k^h} = \frac{1}{s_{k-1}^h} - \frac{1}{\bar{s}_{k-1}^h} + \frac{h_{k-1}}{\tilde{s}_{k-1}^h} - \frac{h_k\varepsilon}{\tilde{s}_k^h} \ge \frac{2^{k-1}n}{\varepsilon^{k-2}\delta} - \frac{1}{1 - 2\varepsilon - \frac{1}{2^{k-1}n}\varepsilon^{k-2}\delta} + \frac{h_{k-1}}{d+1} - \frac{h_k\varepsilon}{d} \ge \ell_{k-1}.$$

By Corollary 3.4, this implies

$$\frac{\varepsilon}{s_k^h} + \frac{\varepsilon}{\bar{s}_k^h} \ge \frac{2^k n}{\varepsilon^{k-2}\delta},$$

which further implies either, since $\frac{1}{n} \le t_k$,

$$x_k^h \le \varepsilon x_{k-1}^h + \frac{\varepsilon^{k-1}\delta}{2^{k-1}n} < \frac{\varepsilon^{k-1}\delta}{4n}\sum_{j=1}^{k-2}\frac{1}{2^{j-2}} + \frac{\varepsilon^{k-1}\delta}{2^{k-1}n} = \frac{\varepsilon^{k-1}\delta}{4n}\sum_{j=1}^{k-1}\frac{1}{2^{j-2}} \le \frac{\varepsilon^{k-1}\delta}{n} \le t_k\varepsilon^{k-1}\delta,$$

or

$$x_k^h \ge 1 - \varepsilon x_{k-1}^h - \frac{\varepsilon^{k-1}\delta}{2^{k-1}n} > 1 - \frac{\varepsilon^{k-1}\delta}{4n}\sum_{j=1}^{k-2}\frac{1}{2^{j-2}} - \frac{\varepsilon^{k-1}\delta}{2^{k-1}n} = 1 - \frac{\varepsilon^{k-1}\delta}{4n}\sum_{j=1}^{k-1}\frac{1}{2^{j-2}} \ge 1 - t_k\varepsilon^{k-1}\delta.$$

This is impossible because $x_k^h \in \left[t_k\varepsilon^{k-1}\delta,\ 1 - t_k\varepsilon^{k-1}\delta\right]$. □

3.2.2 Proof of Proposition 2.2

By analogy with the unit cube $[0,1]^n$, we denote the vertices of the Klee-Minty cube $\mathcal{C}$ using a subset S of $\{1,\ldots,n\}$. For $S \subseteq \{1,\ldots,n\}$, a vertex v^S of $\mathcal{C}$ is defined by

$$v_1^S = \begin{cases} 1, & \text{if } 1 \in S \\ 0, & \text{otherwise} \end{cases}$$

$$v_k^S = \begin{cases} 1 - \varepsilon v_{k-1}^S, & \text{if } k \in S \\ \varepsilon v_{k-1}^S, & \text{otherwise} \end{cases} \qquad k = 2,\ldots,n.$$

Proposition 3.8. *Given $\varepsilon \le \frac{1}{4}$, $\delta < \varepsilon^{n-1}$, $d \ge n2^{n+1}$ and a positive integer h satisfying $Ah \ge b$, for $k \ne n$, the $(k+1)$-th and k-th coordinates of the analytic center $\chi^h(v_n^S)$ of the v_n^S-level set satisfy*

$$|x_{k+1}^h - v_{k+1}^S| \le t_{k+1}\varepsilon^k\delta \Rightarrow |x_k^h - v_k^S| \le t_k\varepsilon^{k-1}\delta.$$

Proof. Assume to the contrary that the statement is false, i.e., for at least one k smaller than or equal to $n-1$, we have: $|x_{k+1}^h - v_{k+1}^S| \le t_{k+1}\varepsilon^k\delta$ and $|x_k^h - v_k^S| > t_k\varepsilon^{k-1}\delta$. We consider a case by case analysis.

Case 1: $v_k^S = 0$
The inequality $|x_k^h - v_k^S| > t_k\varepsilon^{k-1}\delta$ implies $x_k^h > t_k\varepsilon^{k-1}\delta$ and, since $\varepsilon x_k^h \le x_{k+1}^h \le 1 - \varepsilon x_k^h$, we have

$$t_k\varepsilon^k\delta < x_{k+1}^h < 1 - t_k\varepsilon^k\delta.$$

Since $t_{k+1} < t_k$, this implies $t_{k+1}\varepsilon^k\delta < x_{k+1}^h < 1 - t_{k+1}\varepsilon^k\delta$. This contradicts the inequality $|x_{k+1}^h - v_{k+1}^S| \le t_{k+1}\varepsilon^k\delta$, where $v_{k+1}^S = 0$ or 1 because $v_k^S = 0$.

Case 2: $0 < v_k^S < 1$
The inequality $|x_k^h - v_k^S| > t_k\varepsilon^{k-1}\delta$ implies $x_k^h \in \left]0,\ v_k^S - t_k\varepsilon^{k-1}\delta\right[$ or $x_k^h \in \left]v_k^S + t_k\varepsilon^{k-1}\delta,\ 1\right[$. By $]a,b[$ we denote the open interval between a and b.

Subcase 2.1: $x_k^h \in \left]v_k^S + t_k\varepsilon^{k-1}\delta,\ 1\right[$
Since $\varepsilon x_k^h \le x_{k+1}^h \le 1 - \varepsilon x_k^h$, we have $\varepsilon(v_k^S + t_k\varepsilon^{k-1}\delta) < x_{k+1}^h < 1 - \varepsilon(v_k^S + t_k\varepsilon^{k-1}\delta)$. Since $t_{k+1} < t_k$, this implies $\varepsilon v_k^S + t_{k+1}\varepsilon^k\delta < x_{k+1}^h < 1 - \varepsilon v_k^S - t_{k+1}\varepsilon^k\delta$. This contradicts the inequality $|x_{k+1}^h - v_{k+1}^S| \le t_{k+1}\varepsilon^k\delta$, where $v_{k+1}^S = \varepsilon v_k^S$ or $1 - \varepsilon v_k^S$.

Subcase 2.2: $x_k^h \in \left]0,\ v_k^S - t_k\varepsilon^{k-1}\delta\right[$

Subsubcase 2.2.1: $x_k^h < \varepsilon^{k-1} - t_k\varepsilon^{k-1}\delta$
Considering the k-th equation of (2) and successively using $x_k < \varepsilon^{k-1}$ and Corollary 3.4, we have

$$\frac{\varepsilon}{s_{k+1}^h} + \frac{\varepsilon}{\bar{s}_{k+1}^h} = \frac{1}{s_k^h} - \frac{1}{\bar{s}_k^h} + \frac{h_k}{\tilde{s}_k^h} - \frac{h_{k+1}\varepsilon}{\tilde{s}_{k+1}^h} \ge \frac{1}{\varepsilon^{k-1}} - \frac{1}{1-\varepsilon^{k-1}-\varepsilon} + \frac{h_k}{d+1} - \frac{h_{k+1}\varepsilon}{d} \ge \ell_k \ge \frac{2^{k+1}n}{\varepsilon^{k-1}\delta},$$

which implies either

$$x_{k+1}^h \le \varepsilon x_k^h + \frac{1}{2^k n}\varepsilon^k\delta < \varepsilon(\varepsilon^{k-1} - t_k\varepsilon^{k-1}\delta) + \frac{1}{2^k n}\varepsilon^k\delta \le \varepsilon^k - t_{k+1}\varepsilon^k\delta,$$

or

$$x^h_{k+1} \geq 1 - \varepsilon x^h_k - \frac{\varepsilon^k \delta}{2^k n} > 1 - \varepsilon(\varepsilon^{k-1} - t_k \varepsilon^{k-1}\delta) - \frac{\varepsilon^k \delta}{2^k n} \geq 1 - \varepsilon^k + t_{k+1}\varepsilon^k \delta.$$

This contradicts the inequality $|x^h_{k+1} - v^S_{k+1}| \leq t_{k+1}\varepsilon^k \delta$, where $v^S_{k+1} \geq \varepsilon^k$ because $v^S_k > 0$.

Subsubcase 2.2.2: $\varepsilon^{k-1} - t_k \varepsilon^{k-1}\delta \leq x^h_k < v^S_k - t_k \varepsilon^{k-1}\delta$ (*we have* $k \neq 1$ *since* $0 < v^S_k < 1$)
By Lemma 3.6, there is a $\hat{k}$ smaller than or equal to $k-1$ such that $x^h_{\hat{k}} \geq 1 - \varepsilon$, which implies: $s^h_{\hat{k}} \geq 1 - 2\varepsilon$. Considering the $\hat{k}$-th equation of (2) and using $s^h_{\hat{k}} \geq 1 - 2\varepsilon$, we have

$$\frac{\varepsilon^{\hat{k}}}{s^h_{\hat{k}+1}} = \frac{\varepsilon^{\hat{k}-1}}{s^h_{\hat{k}}} - \frac{\varepsilon^{\hat{k}-1}}{\bar{s}^h_{\hat{k}}} - \frac{\varepsilon^{\hat{k}}}{\bar{s}^h_{\hat{k}+1}} + \frac{h_{\hat{k}}\varepsilon^{\hat{k}-1}}{\tilde{s}^h_{\hat{k}}} - \frac{h_{\hat{k}+1}\varepsilon^{\hat{k}}}{\tilde{s}^h_{\hat{k}+1}} \leq \frac{\varepsilon^{\hat{k}-1}}{1-2\varepsilon} - \frac{\varepsilon^{\hat{k}-1}}{\varepsilon} - \frac{\varepsilon^{\hat{k}}}{\bar{s}^h_{\hat{k}+1}} + \frac{h_{\hat{k}}\varepsilon^{\hat{k}-1}}{d} - \frac{h_{\hat{k}+1}\varepsilon^{\hat{k}}}{d+1},$$

which implies

$$\frac{\varepsilon^{\hat{k}}}{s^h_{\hat{k}+1}} \leq \frac{h_{\hat{k}}\varepsilon^{\hat{k}-1}}{d} - \frac{h_{\hat{k}+1}\varepsilon^{\hat{k}}}{d+1} = u_{\hat{k}}\varepsilon^{\hat{k}-1}.$$

The previous inequality, which corresponds to the case $i = \hat{k}$, can be generalized to:

$$\frac{\varepsilon^i}{s^h_{i+1}} \leq \sum_{j=\hat{k}}^{i} u_j \varepsilon^{j-1} \quad \text{for } i = \hat{k}, \dots, k-1,$$

which clearly holds for $k = \hat{k} + 1$ and, for $k > \hat{k} + 1$, is obtained by multiplying the i-th equation of (2) by ε^{i-1} for $i = \hat{k} + 1, \dots, k-1$ and using successively a similar argument. Noticing that we could have initially permute $s^h_{\hat{k}+1}$ and $\bar{s}^h_{\hat{k}+1}$, gives

$$\frac{\varepsilon^{k-1}}{\bar{s}^h_k} \leq \sum_{j=\hat{k}}^{k-1} u_j \varepsilon^{j-1}.$$

Together with the k-th equation of (2), this implies

$$\frac{\varepsilon^k}{s^h_{k+1}} + \frac{\varepsilon^k}{\bar{s}^h_{k+1}} = \frac{\varepsilon^{k-1}}{s^h_k} - \frac{\varepsilon^{k-1}}{\bar{s}^h_k} + \frac{h_k \varepsilon^{k-1}}{\tilde{s}^h_k} - \frac{h_{k+1}\varepsilon^k}{\tilde{s}^h_{k+1}} \geq \frac{h_k \varepsilon^{k-1}}{d+1} - \frac{h_{k+1}\varepsilon^k}{d} - \sum_{j=\hat{k}}^{k-1} u_j \varepsilon^{j-1},$$

i.e.,

$$\frac{\varepsilon^k}{s^h_{k+1}} + \frac{\varepsilon^k}{\bar{s}^h_{k+1}} \geq \ell_k \varepsilon^{k-1} - \sum_{j=\hat{k}}^{k-1} u_j \varepsilon^{j-1}.$$

Using Lemma 3.3, Corollary 3.4 and $u_k \geq \ell_k$, this implies

$$\frac{\varepsilon^k}{s^h_{k+1}} + \frac{\varepsilon^k}{\bar{s}^h_{k+1}} \geq \frac{4n}{\delta} + \sum_{j=1}^{\hat{k}-1} u_j \varepsilon^{j-1} \geq \frac{2^{\hat{k}+1} n}{\delta},$$

which implies either

$$x_{k+1}^h \leq \varepsilon x_k^h + \frac{\varepsilon^k \delta}{2^{\hat{k}} n} < \varepsilon(v_k^S - t_k \varepsilon^{k-1}\delta) + \frac{\varepsilon^k \delta}{2^{\hat{k}} n} \leq \varepsilon v_k^S - t_{k+1}\varepsilon^k \delta,$$

or

$$x_{k+1}^h \geq 1 - \varepsilon x_k^h - \frac{\varepsilon^k \delta}{2^{\hat{k}} n} > 1 - \varepsilon(v_k^S - t_k \varepsilon^{k-1}\delta) - \frac{\varepsilon^k \delta}{2^{\hat{k}} n} \geq 1 - \varepsilon v_k^S + t_{k+1}\varepsilon^k \delta.$$

This contradicts the inequality $|x_{k+1}^h - v_{k+1}^S| \leq t_{k+1}\varepsilon^k \delta$, where $v_{k+1}^S = \varepsilon v_k^S$ or $1 - \varepsilon v_k^S$.

Case 3: $v_k^S = 1$
The inequality $|x_k^h - v_k^S| > t_k \varepsilon^{k-1}\delta$ implies $x_k^h < 1 - t_k \varepsilon^{k-1}\delta$.

Subcase 3.1: $x_k^h < t_k \varepsilon^{k-1}\delta$
Considering the k-th equation of (2) and successively using $x_k^h < t_k \varepsilon^{k-1}\delta$, $t_k \varepsilon^{k-1}\delta \leq \varepsilon$ and Corollary 3.4, we have

$$\frac{\varepsilon}{s_{k+1}^h} + \frac{\varepsilon}{\bar{s}_{k+1}^h} = \frac{1}{s_k^h} - \frac{1}{\bar{s}_k^h} + \frac{h_k}{\tilde{s}_k^h} - \frac{h_{k+1}\varepsilon}{\tilde{s}_{k+1}^h} \geq \frac{1}{t_k \varepsilon^{k-1}\delta} - \frac{1}{1 - t_k \varepsilon^{k-1}\delta - \varepsilon} + \frac{h_k}{d+1} - \frac{h_{k+1}\varepsilon}{d} \geq \ell_k \geq \frac{2^{k+1} n}{\varepsilon^{k-1}\delta},$$

which implies either

$$x_{k+1}^h \leq \varepsilon x_k^h + \frac{\varepsilon^k \delta}{2^k n} < t_k \varepsilon^k \delta + \frac{\varepsilon^k \delta}{2^k n} = (t_k + \frac{1}{2^k n})\varepsilon^k \delta,$$

or

$$x_{k+1}^h \geq 1 - \varepsilon x_k^h - \frac{\varepsilon^k \delta}{2^k n} > 1 - t_k \varepsilon^k \delta - \frac{\varepsilon^k \delta}{2^k n} = 1 - (t_k + \frac{1}{2^k n})\varepsilon^k \delta.$$

This contradicts the inequality $|x_{k+1}^h - v_{k+1}^S| \leq t_{k+1}\varepsilon^k \delta$, where $v_{k+1}^S = \varepsilon$ or $1 - \varepsilon$ because $v_k^S = 1$.

Subcase 3.2: $t_k \varepsilon^{k-1}\delta \leq x_k^h < 1 - t_k \varepsilon^{k-1}\delta$
Subsubcase 3.2.1: $k = 1$
From the first equation of (2), we have

$$\frac{\varepsilon}{s_2^h} + \frac{\varepsilon}{\bar{s}_2^h} = \frac{1}{s_1^h} - \frac{1}{\bar{s}_1^h} + \frac{h_1}{\tilde{s}_1^h} - \frac{h_2 \varepsilon}{\tilde{s}_2^h} \geq -\frac{1}{\delta} + \ell_1 \geq \frac{4n-1}{\delta}.$$

which implies either

$$x_2^h \leq \varepsilon x_1^h + \frac{2\varepsilon\delta}{4n-1} < \varepsilon(1-\delta) + \frac{2\varepsilon\delta}{4n-1} \leq \varepsilon - t_2 \varepsilon \delta,$$

or

$$x_2^h \geq 1 - \varepsilon x_1^h - \frac{2\varepsilon\delta}{4n-1} > 1 - \varepsilon(1-\delta) - \frac{2\varepsilon^k\delta}{4n-1} \geq 1 - \varepsilon + t_2 \varepsilon \delta.$$

This contradicts $|x_2^h - v_2^S| \leq t_2 \varepsilon \delta$ where $v_2^S = \varepsilon$ or $1 - \varepsilon$ since $v_1^S = 1$.

Subsubcase 3.2.2: $k \neq 1$
By Lemma 3.7, there is a $\hat{k}$ smaller than or equal to $k-1$ such that $s^h_{\hat{k}} \geq \frac{\varepsilon^{\hat{k}-1}\delta}{2^{\hat{k}}n}$. Considering the $\hat{k}$-th equation of (2), we have

$$\frac{\varepsilon^{\hat{k}}}{s^h_{\hat{k}+1}} = \frac{\varepsilon^{\hat{k}-1}}{s^h_{\hat{k}}} - \frac{\varepsilon^{\hat{k}-1}}{\bar{s}^h_{\hat{k}}} - \frac{\varepsilon^{\hat{k}}}{\bar{s}^h_{\hat{k}+1}} + \frac{h_{\hat{k}}\varepsilon^{\hat{k}-1}}{\tilde{s}^h_{\hat{k}}} - \frac{h_{\hat{k}+1}\varepsilon^{\hat{k}}}{\tilde{s}^h_{\hat{k}+1}} \leq \frac{2^{\hat{k}}n}{\delta} + \frac{h_{\hat{k}}\varepsilon^{\hat{k}-1}}{d} - \frac{h_{\hat{k}+1}\varepsilon^{\hat{k}}}{d+1} = u_{\hat{k}}\varepsilon^{\hat{k}-1} + \frac{2^{\hat{k}}n}{\delta}.$$

This inequality, which corresponds to the case $i = \hat{k}$, can be generalized to:

$$\frac{\varepsilon^i}{s^h_{i+1}} \leq \sum_{j=\hat{k}}^{i} u_j \varepsilon^{j-1} + \frac{2^{\hat{k}}n}{\delta} \quad \text{ for } i = \hat{k}, \ldots, k-1,$$

which clearly holds for $k = \hat{k}+1$ and, for $k > \hat{k}+1$, is obtained by multiplying the i-th equation of (2) by ε^{i-1} for $i = \hat{k}+1, \ldots, k-1$ and using successively a similar argument. Noticing that we could have initially permute $s^h_{\hat{k}+1}$ and $\bar{s}^h_{\hat{k}+1}$, gives

$$\frac{\varepsilon^{k-1}}{\bar{s}^h_k} \leq \sum_{j=\hat{k}}^{k-1} u_j \varepsilon^{j-1} + \frac{2^{\hat{k}}n}{\delta}.$$

Together with the k-th equation of (2), this implies

$$\frac{\varepsilon^k}{s^h_{k+1}} + \frac{\varepsilon^k}{\bar{s}^h_{k+1}} = \frac{\varepsilon^{k-1}}{s^h_k} - \frac{\varepsilon^{k-1}}{\bar{s}^h_k} + \frac{h_k\varepsilon^{k-1}}{\tilde{s}^h_k} - \frac{h_{k+1}\varepsilon^k}{\tilde{s}^h_{k+1}} \geq -\sum_{i=\hat{k}}^{k-1} u_i\varepsilon^{i-1} - \frac{2^{\hat{k}}n}{\delta} + \frac{h_k\varepsilon^{k-1}}{d+1} - \frac{h_{k+1}\varepsilon^k}{d}.$$

Using Lemma 3.3, Corollary 3.4, and $u_k \geq \ell_k$, the previous inequality gives

$$\frac{\varepsilon^k}{s^h_{k+1}} + \frac{\varepsilon^k}{\bar{s}^h_{k+1}} \geq \ell_k\varepsilon^{k-1} - \sum_{i=\hat{k}}^{k-1} u_i\varepsilon^{i-1} - \frac{2^{\hat{k}}n}{\delta} \geq \frac{4n}{\delta} + \sum_{i=1}^{\hat{k}-1} u_i\varepsilon^{i-1} - \frac{2^{\hat{k}}n}{\delta} \geq \frac{2^{\hat{k}+1}n}{\delta} - \frac{2^{\hat{k}}n}{\delta} = \frac{2^{\hat{k}}n}{\delta},$$

which implies either

$$x^h_{k+1} \leq \varepsilon x^h_k + \frac{\varepsilon^k\delta}{2^{\hat{k}-1}n} < \varepsilon(1 - t_k\varepsilon^{k-1}\delta) + \frac{\varepsilon^k\delta}{2^{\hat{k}-1}n} \leq \varepsilon - t_{k+1}\varepsilon^k\delta,$$

or

$$x^h_{k+1} \geq 1 - \varepsilon x^h_k - \frac{\varepsilon^k\delta}{2^{\hat{k}-1}n} > 1 - \varepsilon(1 - t_k\varepsilon^{k-1}\delta) - \frac{\varepsilon^k\delta}{2^{\hat{k}-1}n} \geq 1 - \varepsilon + t_{k+1}\varepsilon^k\delta.$$

This contradicts $|x^h_{k+1} - v^S_{k+1}| \leq t_{k+1}\varepsilon^k\delta$ where $v^S_{k+1} = \varepsilon$ or $1-\varepsilon$ since $v^S_k = 1$. □

Proposition 2.2 is a direct corollary of Proposition 3.8 since, for $S \neq \emptyset$ and $S \neq \{n\}$, we have $|x^h_n - v^S_n| = 0$; implying $|x^h_k - v^S_k| \leq t_k\varepsilon^{k-1}\delta$ for $k = 1, \ldots, n-1$. In other words, $|\chi^h(v^S_n) - v^S|_\infty \leq \delta$. Furthermore, by Corollary 3.5 we have $|\chi^h - v^{\{n\}}|_\infty \leq \delta$, and the central path converges to the origin $v^\emptyset$. □

4 Remarks and Future Work

- We showed that, without changing the geometry of the feasible set of KM, the central path can be forced to visit arbitrarily small neighborhoods of all the vertices of the Klee-Minty n-cube by carefully adding redundant constraints.

- This result highlights that, although the central path is a smooth analytical curve in the interior of the set of feasible solutions, it might be severely distracted by redundant constraints. In particular, exponentially many redundant constraints inter-playing with the geometry of the problem, may force the central path to take exponentially many and arbitrarily sharp turns.

- Our example leads to an $\Omega(2^n)$ lower bound for the number of iterations needed for central path-following interior point methods. The theoretical iteration-complexity upper bound $O(\sqrt{N}L) = O(2^{9n}n^4)$ as, for this example, the number of constraints $N = O(2^{6n}n^2)$ and the bit-length of the input-data $L = O(2^{6n}n^3)$. Therefore, the $\Omega(2^n)$ lower bound yields an $\Omega(\sqrt[6]{\frac{N}{\ln^2 N}})$ iteration-complexity lower bound. Using a different analysis, Todd and Ye [10] gave an $\Omega(\sqrt[3]{N})$ iteration-complexity lower bound between two updates of the barrier function. In a subsequent paper, Deza, Nematollahi and Terlaky [2] essentially closed the gap between the lower and upper bounds.

- State-of-the-art preprocessing tools in modern linear optimization software would eliminate the added redundant inequalities. Therefore, interior point methods based codes would solve the preprocessed KM^h efficiently, just as commercial simplex codes do solve the KM in only one pivot. A challenging task would be to design a variant of KM^h that cannot be easily simplified by known preprocessing heuristics.

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