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Advanced Optimization Laboratory

Title:

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AdvOL-Report No. 2007/9

July, 2007
Hamilton, Ontario, Canada

Noname manuscript No.
(will be inserted by the editor)

New stopping criteria for detecting infeasibility in conic optimization

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Received: July 17, 2007/ Accepted: date

Abstract Detecting infeasibility in conic optimization and providing certificates for infeasibility pose a bigger challenge than in the linear case due to the lack of strong duality. In this paper we generalize the approximate Farkas lemma of Todd and Ye [9] from the linear to the general conic setting, and use it to propose stopping criteria for interior point algorithms using self-dual embedding. The new criteria can identify if the solutions have large norm, thus they give an indication of infeasibility. The modified algorithms enjoy the same complexity bounds as the original ones, without assuming that the problem is feasible. Issues about the practical application of the criteria are also discussed.

Keywords approximate Farkas theorem · stopping criteria · conic optimization · infeasibility

Mathematics Subject Classification (2000) 90C51, 90C46

1 Introduction

Identifying infeasibility in conic optimization is of key importance. An infeasible model usually does not yield valuable information, it will have to be corrected and resolved. Thus, early identification of infeasibility saves computing time.

This paper presents some new stopping criteria for interior point methods for conic optimization. The results are based on an approximate theorem of the alternatives. First we briefly review the duality theory of conic optimization, for more details see [1].

Let $\mathscr{K} \subset \mathbb{R}^n$ be a closed, convex, pointed cone with nonempty interior and let $\mathscr{K}^* = \left\{y \in \mathbb{R}^n : x^T y \geq 0 \; \forall x \in \mathscr{K}\right\}$ be the dual cone associated with $\mathscr{K}$. Then $\mathscr{K}^*$ possesses the same properties as $\mathscr{K}$, and $\mathscr{K}^{**} = \mathscr{K}$. Now we can define the generalized inequalities $\succeq_{\mathscr{K}}$ and $\succ_{\mathscr{K}}$ for which $x \succeq_{\mathscr{K}} y \Leftrightarrow x - y \in \mathscr{K}$ and $x \succ_{\mathscr{K}} y \Leftrightarrow x - y \in \operatorname{int}(\mathscr{K})$. Consider now the

The authors were supported by the Canada Research Chairs program, NSERC (Discovery Grant #5-48923) and MITACS.

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following primal dual pair of conic optimization problems:

$$\begin{aligned} \min\ & c^T x \\ Ax &= b \\ x &\succeq_{\mathcal{K}} 0 \end{aligned} \tag{Cone$_\mathrm{P}$}$$

and

$$\begin{aligned} \max\ & b^T y \\ A^T y + s &= c \\ s &\succeq_{\mathcal{K}^*} 0. \end{aligned} \tag{Cone$_\mathrm{D}$}$$

where $x, s, c \in \mathbb{R}^n$, $A \in \mathbb{R}^{m\times n}$, $b, y \in \mathbb{R}^m$ and $\mathcal{K} \subset \mathbb{R}^{m_i}$ is a cone with the above properties. Furthermore, we will assume that the rows of matrix A are linearly independent. We can state the weak duality theorem of conic optimization:

Theorem 1.1 (Weak duality) *If x is a feasible solution for* (Cone$_\mathrm{P}$) *and y is a feasible solution for* (Cone$_\mathrm{D}$) *then $c^T x \ge b^T y$.*

A conic problem is strictly feasible, if there is a feasible solution that satisfies the strict version of all the inequalities. Now we are ready to state the strong duality theorem:

Theorem 1.2 (Strong duality) *If one of the problems* (Cone$_\mathrm{P}$) *and* (Cone$_\mathrm{D}$) *is strictly feasible and its objective values are bounded then the other problem is solvable and the optimal values are equal. If both problems are strictly feasible then both are solvable and the optimal values are equal.*

Using the strong duality result one can easily prove the appropriate version of Farkas lemma.

Theorem 1.3 (Farkas theorem, conic case) *Consider the following two systems:*

$$\begin{aligned} Ax &= b \\ x &\succeq_{\mathcal{K}} 0 \end{aligned} \tag{ConFeas$_\mathrm{P}$}$$

and

$$\begin{aligned} A^T y &\preceq_{\mathcal{K}^*} 0 \\ b^T y &= 1. \end{aligned} \tag{ConFeas$_\mathrm{D}$}$$

Then (ConFeas$_\mathrm{D}$) *is not solvable if and only if* (ConFeas$_\mathrm{P}$) *is almost solvable, i.e., if for every $\varepsilon > 0$ there exists b' such that $\|b - b'\| \le \varepsilon$ and the perturbed system $Ax = b'$, $x \succeq_{\mathcal{K}} 0$ is solvable.*

This theorem provides a tool to prove infeasibility: a solution for (ConFeas$_\mathrm{P}$) certifies that (ConFeas$_\mathrm{D}$) is not solvable. The difficulty is that in practice we never get an exact solution for these systems. Whence a more relevant question is: what can we deduce, if anything, from an almost certificate?

2 An approximate Farkas theorem

Let us consider the primal-dual conic optimization problems (Cone_{P}) and (Cone_{D}). Now define some useful quantities.

Definition 2.1 Let $\|.\|$ be any norm on $\mathbb{R}^n$, and let $\|.\|^*$ be its dual norm, i.e., $\|v\|^* = \max\left\{v^T y : \|y\| = 1\right\}$. Now define

$$\begin{aligned} \alpha_x = \quad & \inf \|x\| \\ & Ax = b \\ & x \succeq_{\mathscr{K}} 0, \end{aligned} \tag{P_{α_x}}$$

and

$$\begin{aligned} \beta_u = \quad & \inf \|u\|^* \\ & A^T y \preceq_{\mathscr{K}^*} u \\ & b^T y = 1. \end{aligned} \tag{D_{β_u}}$$

In other words, α_x is the norm of the smallest solution to (Cone_{P}) and β_y is the norm of the smallest dual improving direction.

The results of this section are based on the following theorem, first mentioned in [8]:

Theorem 2.1 (Approximate Farkas theorem for conic problems) *Assuming the convention* $0 \cdot \infty = 1$*, we have* $\alpha_x \beta_u = 1$.

Proof This proof is based on an idea of Jos Sturm [8].

If $\alpha_x = 0$ then necessarily $b = 0$, therefore (D_{β_u}) is infeasible and $\beta_u = +\infty$.

If $\alpha_x = +\infty$ then there is no $x \succeq_{\mathscr{K}} 0$ such that $Ax = b$ (i.e., (P_{α_x}) is infeasible) and in this case we can apply the Farkas theorem (Thm. 1.3) to deduce that (D_{β_u}) is almost feasible, i.e., for every $\varepsilon > 0$ there is y_ε and u_ε such that $A^T y_\varepsilon \preceq_{\mathscr{K}^*} u_\varepsilon$, $b^T y_\varepsilon = 1$ and $\|u_\varepsilon\|^* < \varepsilon$, hence $\beta_u = 0$.

Now we can assume that $0 < \alpha_x < +\infty$. Let us introduce the so-called norm-cone:

$$\mathscr{K}_{\text{norm}} := \left\{(x_0, x) \in \mathbb{R}^{n+1} : x_0 \geq \|x\|\right\}. \tag{2.1}$$

It is straightforward to check that this set is indeed a closed, convex, pointed, solid cone. Now α_x can be written as

$$\begin{aligned} \alpha_x = \quad & \inf x_0 \\ & Ax = b \\ & x \succeq_{\mathscr{K}} 0 \\ & (x_0, x) \succeq_{\mathscr{K}_{\text{norm}}} 0. \end{aligned} \tag{P'_{α_x}}$$

Let $\bar{x}$ be a feasible solution for (P_{α_x}). Now $(\|\bar{x}\| + 1, \bar{x})$ provides a strictly feasible solution for (P'_{α_x}) and applying the strong duality theorem we get that

$$\begin{aligned} \alpha_x = \quad & \sup b^T y \\ & \|u\|^* \leq 1 \\ & A^T y \preceq_{\mathscr{K}^*} u. \end{aligned} \tag{D'_{β_u}}$$

As both problem (P'_{α_x}) and (D'_{β_u}) are strictly feasible we know by strong duality that (D'_{β_u}) is solvable. Let $\hat{y}, \hat{u}$ be an optimal solution pair. We can assume that for this solution $\|u\|^* = 1$, since changing u does not change the objective function. Defining $\bar{y} = \hat{y}/\alpha_x$ and $\bar{u} = \hat{u}\alpha_x$ we get

$$\begin{aligned}\|\bar{u}\|^* &= 1/\alpha_x \\ \mathscr{A}^T \bar{y} &\preceq_{\mathscr{K}^*} \bar{u} \\ b^T \bar{y} &= 1,\end{aligned}$$

implying $\beta_u \le 1/\alpha_x$, thus $\alpha_x \beta_u \le 1$.

Finally, if either one of (P_{α_x}) or (D_{β_u}) is not feasible, then the corresponding value is $+\infty$, thus $\alpha_x \beta_u \ge 1$. If both are feasible, then let x and u, y be feasible solutions. In this case:

$$1 = b^T y = \underbrace{x^T}_{\in \mathscr{K}} \underbrace{A^T y}_{\preceq_{\mathscr{K}^*} u} \le x^T u \le \|x\| \, \|u\|^*,$$

yielding $\alpha_x \beta_u \ge 1$. This completes the proof. □

2.1 The geometry of infeasible problems

New let us try to understand the geometry of these problems, especially weak infeasibility. For this we introduce the following quantity:

$$\begin{aligned}\alpha_x^\varepsilon = \quad & \inf \|x\| \\ & Ax = b^\varepsilon \\ & x \succeq_{\mathscr{K}} c^\varepsilon \\ \|b - b^\varepsilon\|^* &\le \varepsilon \\ \|c^\varepsilon\|^* &\le \varepsilon\end{aligned} \tag{P_ε}$$

In this notation our previous α_x is α_x^0, i.e., the norm of the smallest solution for the unperturbed system. It is obvious that if $\varepsilon_1 \le \varepsilon_2$ then $\alpha_x^{\varepsilon_1} \ge \alpha_x^{\varepsilon_2}$, since the infimum is taken over a smaller set. This means that α_x^ε converges as $\varepsilon \to 0$, allowing the limit to be infinity. Let the limit be denoted by $\bar{\alpha}_x$. The first interesting question is how it relates to $\alpha_x = \alpha_x^0$:

Theorem 2.2 *$\bar{\alpha}_x = \alpha_x^0$, thus the function $\varepsilon \mapsto \alpha_x^\varepsilon$ is continuous at $\varepsilon = 0$.*

Proof First, let us notice that since α_x^ε is increasing in ε, $\alpha_x^0 \ge \alpha_x^\varepsilon$ for all $\varepsilon \ge 0$, and taking the limit as $\varepsilon \to 0$ we have $\bar{\alpha}_x \le \alpha_x^0$. So it remains to prove the opposite inequality, i.e., $\bar{\alpha}_x \ge \alpha_x^0$.

In the case when $\bar{\alpha}_x = +\infty$ the inequality is trivial, so we can restrict ourselves to $\bar{\alpha}_x$ being finite. Since α_x^ε is defined as an infimum the following system of (conic) inequalities must be solvable for all positive ε and arbitrary n.

$$\begin{aligned}Ax_n^\varepsilon &= b_n^\varepsilon \\ x_n^\varepsilon &\succeq_{\mathscr{K}} c_n^\varepsilon \\ \|b - b_n^\varepsilon\|^* &\le \varepsilon \\ \|c_n^\varepsilon\|^* &\le \varepsilon \\ \|x_n^\varepsilon\| &\le \alpha_x^\varepsilon + \frac{1}{n}\end{aligned}$$

Let us fix $\varepsilon > 0$. It is clear, that

$$\forall n: \ \|x_n^\varepsilon\| \le \alpha_x^\varepsilon + 1, \quad \|b - b_n^\varepsilon\|^* \le \varepsilon, \quad \|c_n^\varepsilon\|^* \le \varepsilon.$$

Using the fact that $\bar{\alpha}_x$ is finite we have that α_x^ε is finite for all $\varepsilon > 0$. This means that the solutions $(x_n^\varepsilon, b_n^\varepsilon, c_n^\varepsilon)$ are in a compact set for all n. Thus, there is a convergent subsequence such that

$$x_{n_k}^\varepsilon \to x^\varepsilon, \quad b_{n_k}^\varepsilon \to b^\varepsilon, \quad c_{n_k}^\varepsilon \to c^\varepsilon,$$

as $k \to \infty$. The limit points satisfy the system

$$\begin{aligned} Ax^\varepsilon &= b^\varepsilon \\ x^\varepsilon &\succeq_{\mathscr{K}} c^\varepsilon \\ \|b - b^\varepsilon\|^* &\le \varepsilon \\ \|c^\varepsilon\|^* &\le \varepsilon \\ \|x^\varepsilon\| &\le \alpha_x^\varepsilon \end{aligned}$$

which means that they form a feasible solution of (P_ε) with the additional property that $\|x^\varepsilon\| \le \alpha_x^\varepsilon$. These two facts imply that $\|x^\varepsilon\| = \alpha_x^\varepsilon$, i.e., the infimum in (P_ε) is attained, so it is in fact a minimum.

Now we have $\|x^\varepsilon\| = \alpha_x^\varepsilon \le \bar{\alpha}_x < +\infty$, so there is a sequence $\varepsilon_k \to 0$ that $x^{\varepsilon_k} \to x$, $b^{\varepsilon_k} \to b$, and $c^{\varepsilon_k} \to 0$ as $k \to \infty$. Taking the limit of the inequalities we have that

$$\begin{aligned} \|x\| &\le \bar{\alpha}_x \\ Ax &= b \\ x &\succeq_{\mathscr{K}} 0 \end{aligned}$$

Here the last two equations imply that x is a feasible solution of (P_ε) with $\varepsilon = 0$, so

$$\alpha_x^0 \le \|x\| \le \bar{\alpha}_x.$$

This completes the proof. □

From the last part of the proof we can draw a useful corollary:

Corollary 2.1 *If $\alpha_x < \infty$ then there exists an $x \in \mathbb{R}^n$ such that $\alpha_x = \alpha_x^0 = \|x\|$. In other words, the infimum in (P_{α_x}) is attained.*

In the general conic case α_x alone does not tell everything about the infeasibility of the primal system (Cone$_\mathrm{P}$), the further distinction is made by α_x^ε. If $\alpha_x = \infty$ then (Cone$_\mathrm{P}$) is clearly infeasible, but if now α_x^ε is finite for every $\varepsilon > 0$ then (Cone$_\mathrm{P}$) is only weakly infeasible. Similarly, if there is an $\varepsilon > 0$ such that $\alpha_x^\varepsilon = \infty$ then the problem is strongly infeasible, meaning that slightly perturbing the problem does not make it feasible. Similar results can be proved for the dual case.

3 Stopping criteria for self-dual models

In this section we derive two sets of stopping criteria for the homogeneous self-dual model for conic optimization.

3.1 Homogeneous self-dual model for conic optimization

This model has already been described and analyzed in many works, see [3,4]. Given the primal-dual pair (Cone_P) and (Cone_D) consider the following system:

$$
\begin{array}{rrrrrl}
 & & & \min\ (\bar{x}^T\bar{s}+1)\theta & & \\
 & Ax & -b\tau & +\bar{b}\theta & & = 0 \\
-A^T y & & +c\tau & -\bar{c}\theta & -s & = 0 \\
b^T y & -c^T x & & +\bar{z}\theta & -\kappa & = 0 \\
-\bar{b}^T y & -\bar{c}^T x & -\bar{z}^T\tau & & & = -\bar{x}^T\bar{s}-1 \\
 & x \succeq_{\mathcal{K}} 0, & \tau \ge 0, & & s \succeq_{\mathcal{K}^*} 0, & \kappa \ge 0,
\end{array}
\tag{HSD}
$$

where $\bar{x},\bar{s} \in \mathbb{R}^n$, $\bar{y} \in \mathbb{R}^m$ are arbitrary starting points, τ, θ are scalars, $\bar{b} = b - A\bar{x}$, $\bar{c} = c - A^T\bar{y} - \bar{s}$ and $\bar{z} = c^T\bar{x} - b^T\bar{y} + 1$. This model has the following properties.

Theorem 3.1 (Properties of the HSD model) *System* (HSD) *is self-dual and it has a strictly feasible starting point, namely* $(x,s,y,\tau,\theta,\kappa) = (\bar{x},\bar{s},\bar{y},1,1,0)$. *The optimal value of these problems is* $\theta = 0$*, and if* $\tau > 0$ *at optimality then* $(x/\tau, y/\tau, s/\tau)$ *is an optimal solution for the original primal-dual problem. If* $\tau = 0$*, then the problem is either unbounded, infeasible, or the duality gap at optimality is nonzero.*

Let us solve the homogeneous problem with a path-following interior point method that generates a sequence of iterates $\left(x^{(k)}, s^{(k)}, y^{(k)}, \tau^{(k)}, \theta^{(k)}, \kappa^{(k)}\right)$, such that

$$\tau^{(k)}\kappa^{(k)} \ge (1-\beta)\theta^{(k)}, \tag{3.1}$$

where β is a fixed constant. This is a standard assumption for interior point methods, see [5] for details.

3.2 New stopping criteria

In practice we have to live with the fact that we never have exact optimal solutions. Similarly, we can rarely guarantee infeasibility, but we can certify that every feasible solution should have a large norm. If the threshold is set high enough then this is sufficient in most cases. The results of these sections are generalizations of [9].

Let ρ be a large enough number (about 10^9 or more in general) and ε a small number (typically 10^{-9}). The first stopping criterion deals with optimality, it is activated if the current iterate provides an ε-optimal and ε-feasible solution:

$$
\begin{aligned}
\|Ax - b\tau\|^* &\le \varepsilon\tau \\
\left\|A^T y + s - c\tau\right\|^* &\le \varepsilon\tau \\
c^T x - b^T y &\le \varepsilon\tau.
\end{aligned}
\tag{R1}
$$

As we mentioned earlier, if $\tau = 0$ is an optimal solution for the self-dual model then the original problems do not have an optimal solution with zero duality gap. A small value of τ gives us slightly less, that is our second criterion:

$$\tau \le \frac{1-\beta}{1+\rho}. \tag{R2}$$

We have the following result:

Lemma 3.1 (Identification of large optimal solutions) *If stopping rule* (R2) *is satisfied then for every optimal solution x^*, (y^*, s^*) of the original primal-dual problem we have $x^{*T}\bar{s} + s^{*T}\bar{x} \geq \rho$, where $\bar{x}$ and $\bar{s}$ are the starting points of the HSD model.*

Proof Assume that the stopping rule is activated, let x^*, (y^*, s^*) be an optimal solution of the original primal-dual problem, and assume that they have small norm, i.e., $x^{*T}\bar{s} + s^{*T}\bar{x} < \rho$. Let

$$\alpha = \frac{\bar{x}^T\bar{s} + 1}{x^{*T}\bar{s} + s^{*T}\bar{x} + 1} > 0, \tag{3.2}$$

and define $\left(\bar{x}, \bar{y}, \bar{s}, \bar{\tau}, \bar{\theta}, \bar{\kappa}\right) = (\alpha x^*, \alpha y^*, \alpha s^*, \alpha, 0, 0)$. This tuple forms an optimal solution for the self-dual system (HSD). Now let us subtract the two self dual systems:

$$\begin{array}{rrrrl} & A(x-\bar{x}) & -b(\tau-\bar{\tau}) & +\bar{b}(\theta-\bar{\theta}) & = 0 \\ -A^T(y-\bar{y}) & & +c(\tau-\bar{\tau}) & -\bar{c}(\theta-\bar{\theta}) & = s-\bar{s} \\ b^T(y-\bar{y}) & -c^T(x-\bar{x}) & & +\bar{z}(\theta-\bar{\theta}) & = \kappa-\bar{\kappa} \\ -\bar{b}^T(y-\bar{y}) & -\bar{c}^T(x-\bar{x}) & -\bar{z}^T(\tau-\bar{\tau}) & & = 0. \end{array} \tag{3.3}$$

Premultiplying this system by $(y-\bar{y}, x-\bar{x}, \tau-\bar{\tau}, \theta-\bar{\theta})$ gives

$$0 = (x-\bar{x})^T(s-\bar{s}) + (\tau-\bar{\tau})(\kappa-\bar{\kappa}). \tag{3.4}$$

Rearranging the terms and using that $x^T s + \tau\kappa = \theta(1 - \bar{x}^T\bar{s})$ we get

$$x^T\bar{s} + s^T\bar{x} + \tau\bar{\kappa} + \kappa\bar{\tau} = \left(\bar{x}^T s^* + \bar{s}^T x^* + 1\right)\theta. \tag{3.5}$$

Now

$$\tau = \tau\frac{x^T\bar{s} + s^T\bar{x} + \tau\bar{\kappa} + \kappa\bar{\tau}}{(\bar{x}^T s^* + \bar{s}^T x^* + 1)\,\theta} \geq \tau\frac{\kappa\bar{\tau}}{(\bar{x}^T s^* + \bar{s}^T x^* + 1)\,\theta} \geq \frac{(1-\beta)\bar{\tau}}{\bar{x}^T s^* + \bar{s}^T x^* + 1} > \frac{1-\beta}{1+\rho}, \tag{3.6}$$

contradicting (R2). This proves the lemma. □

Stopping rule (R2) guarantees that the optimal solutions have large norm. There might be some moderate-sized feasible solutions, but they are far from optimal. This usually suggests some modelling error or unboundedness.

The third set of criteria identifies large feasible solutions.

$$b^T y \geq (\tau\,\|c\|^* + \theta\,\|\bar{c}\|^*)\,\bar{\rho} \tag{R3a}$$

$$c^T x \leq -\left(\tau\,\|b\|^* + \theta\,\left\|\bar{b}\right\|^*\right)\bar{\rho} \tag{R3b}$$

Lemma 3.2 (Identification of large feasible solutions) *If stopping criterion* (R3a) *is activated then for every feasible solution x of the primal problem we have $\|x\| \geq \bar{\rho}$. Similarly, if stopping criterion* (R3b) *is activated then for every feasible solution (y, s) of the dual problem we have $\|s\|^* \geq \bar{\rho}$.*

Proof We only prove the first statement as the second one is very similar. Let $\bar{y} = y/b^T y$ and $\bar{u} = (\tau c - \theta\bar{c})/b^T y$, then $\bar{y}$ and $\bar{u}$ are feasible in (D_{β_u}), thus $\beta_u \leq 1/\bar{\rho}$ and by Theorem 2.1 we have $\alpha_x \geq \bar{\rho}$. □

This technique is already used in most conic optimization software packages.

3.3 Complexity of the criteria

Now we show that these conditions are indeed practical in the sense that one of them is activated after a polynomial number of iterations. This is trivial if we know in advance that our original (not the self-dual) problem is feasible. Now let us assume that we use a feasible interior point method to solve the self-dual problem. Such algorithms can produce a solution satisfying $\theta < \varepsilon$ (if such a solution exists) in $\mathscr{O}(\sqrt{\vartheta}\log(1/\varepsilon))$ iterations, where ϑ depends only on $\mathscr{K}$, see, e.g., [5,7]. The question is what happens if this is not the case, i.e., if either the original problem is infeasible or there are no solutions with zero duality gap. First let us see what we get from rules (R1) and (R2).

Theorem 3.2 (Complexity of the IPM with (R1) and (R2)) *Either rule* (R1) *or* (R2) *is activated in*

$$\mathscr{O}\left(\sqrt{\vartheta}\log\left(\frac{\max\left\{\left\|\bar{b}\right\|^*,\|\bar{c}\|^*,\bar{z}\right\}(1+\rho)}{\varepsilon}\right)\right) \tag{3.7}$$

iterations.

Proof Using $Ax - b\tau = -\bar{b}\theta$, $A^Ty + s - c\tau = -\bar{c}\theta$ and $c^Tx - b^Ty = \bar{z}\theta - \kappa$ we get that criterion (R1) is satisfied if

$$\frac{\theta}{\tau} \leq \frac{\varepsilon}{\max\left\{\left\|\bar{b}\right\|^*,\|\bar{c}\|^*,\bar{z}\right\}}. \tag{3.8}$$

Assume that rule (R2) is not activated during the iterations, this means that $\tau > (1-\beta)/(1+\rho)$ for every fixed β. This implies that rule (R1) is satisfied if

$$\theta < \frac{(1-\beta)\varepsilon}{(1+\rho)\max\left\{\left\|\bar{b}\right\|^*,\|\bar{c}\|^*,\bar{z}\right\}}, \tag{3.9}$$

and using assumption (3.1) about the algorithm we get the statement of the theorem. □

Corollary 3.1 *Setting $\rho = 1/\varepsilon$ the complexity of the algorithm with stopping criteria* (R1) *and* (R2) *is $\mathscr{O}(\sqrt{\vartheta}\log(1/\varepsilon))$, the same order as the original algorithm running on a feasible problem. In this many iterations either the algorithm finds an ε-optimal and ε-feasible solution or it proves that the norm of any optimal solution is larger than $1/\varepsilon$.*

The second theorem deals with rules (R1), (R3a) and (R3b). Let ε and ρ be the parameters for accuracy and feasible solution size. Let us define the following quantities:

$$\bar{\rho} = \max\left\{\bar{z},\rho\max\left\{\|c\|^*,\|\bar{c}\|^*,\|b\|^*,\left\|\bar{b}\right\|^*\right\}\right\} \tag{3.10}$$

$$\bar{\varepsilon} = \min\left\{\frac{2}{3},\frac{\varepsilon}{\max\left\{\left\|\bar{b}\right\|^*,\|\bar{c}\|^*,\bar{z}\right\}}\right\} \tag{3.11}$$

Theorem 3.3 (Complexity of the IPM with (R1) and (R3)) *Either rule (R1) or (R3) is activated in not more than*

$$\mathscr{O}\left(\sqrt{\vartheta}\log\frac{\bar{\rho}}{\bar{\varepsilon}}\right) \tag{3.12}$$

iterations.

Proof We prove that if

$$\theta \leq \frac{(1-\beta)\bar{\varepsilon}^2}{4\bar{\rho}} \tag{3.13}$$

then one of the stopping criteria (R1), (R3a), (R3b) is satisfied. Assume to the contrary that none of the three criteria is satisfied. Since (R1) is not active we get

$$\frac{\theta}{\tau} \geq \bar{\varepsilon} \tag{3.14}$$

and combining this with (3.1) we have

$$\kappa \geq (1-\beta)\frac{\theta}{\tau} > (1-\beta)\bar{\varepsilon}. \tag{3.15}$$

Using (3.13) we can continue the estimate:

$$\frac{\kappa}{\theta} > \frac{4\bar{\rho}}{\bar{\varepsilon}} \geq \left(3+\frac{2}{\bar{\varepsilon}}\right)\bar{\rho} \geq \bar{z}+2\left(1+\frac{\tau}{\theta}\right)\bar{\rho} \geq \bar{z}+\frac{\rho}{\theta}\left(\tau\left(\|c\|^*+\|b\|^*\right)+\theta\left(\|\bar{c}\|^*+\left\|\bar{b}\right\|^*\right)\right),$$

where we used that $\bar{\varepsilon} \leq 2/3$, thus $4/\bar{\varepsilon} \geq 3+2/\bar{\varepsilon}$. Since (R3a) and (R3b) are not satisfied we can continue with

$$\frac{\kappa}{\theta} > \bar{z}+\frac{1}{\theta}\left(b^T y - c^T x\right) = \bar{z}+\frac{1}{\theta}(\kappa-\bar{z}\theta) = \frac{\kappa}{\theta}, \tag{3.16}$$

which is a contradiction. The complexity result follows easily, using assumption (3.1) about the algorithm. □

Corollary 3.2 *Again, setting $\rho = 1/\varepsilon$ the complexity of the algorithm with stopping criteria* (R1)*,* (R3a) *and* (R3b) *is $\mathcal{O}\left(\sqrt{\vartheta}\log(1/\varepsilon)\right)$, the same order as the original algorithm running on a feasible problem. In this many iterations either the algorithm finds an ε-optimal solution or it proves that the norm of any feasible solution is larger than $1/\varepsilon$.*

4 Summary and future work

In this paper we presented two new stopping criteria for IPMs for general conic optimization. They can detect if the norm of feasible or optimal solutions is large, thus give an indication of infeasibility. We fully analyzed the complexity of the resulting algorithms and showed that the new criteria do not change the order of the number of iterations. Most of the future questions in this area are about the implementation of the stopping criteria.

4.1 How big is big enough?

In all the criteria discussed in this paper we have a parameter ρ, which is supposed to be big. The role of ρ is the same in all the conditions: if some computed value exceeds ρ, then we declare infeasibility. The natural question is: how big should ρ be? In practice $\rho = 1/\varepsilon$ is a popular choice, with a reasonable justification. Either we find an ε-optimal solution, or we find an ε-certificate for infeasibility. This guarantees a certain primal-dual symmetry.

Another option is to apply a preprocessing phase to obtain bounds on the variables in the problem, and compute ρ accordingly. For very large scale problems with 5000×5000 semidefinite matrices if the elements are in the order of some hundreds then the norm of the matrix is more than 10^9, thus a larger ρ should be chosen. This method is yet to be implemented in major conic optimization solvers.

4.2 Large optimal solutions

The set of stopping criteria that can certify that all the optimal solutions of a problem are large has not been implemented in any conic optimization solver yet. By notifying the user about the problem early on, this method would save computing time.

4.3 Handling weak infeasibility

Weakly infeasible problems are infeasible but there is no certificate that proves that. From another point of view, the problem is infeasible, but perturbing it slightly we can get either a feasible or an infeasible problem. Since practical solvers usually deal with approximate solutions and certificates, weakly infeasible problems are hard (if not practically impossible) to identify. An almost solution and an almost certificate for infeasibility can coexist, and there is no easy way to decide if the problem is in fact feasible or infeasible.

This is exactly how this problem manifests in practice: in the set of stopping criteria more conditions are satisfied simultaneously. Of course, this does not mean weak infeasibility automatically, it only indicates that the problem is close to the borderline between feasible and infeasible problems. We do not have a way to decide which side the problem lies unless we carry out the operations more accurately.

The conclusions we get from the criteria are also twofold. We end up with an approximate solution (usually of large norm) and we also deduce that all feasible or optimal solutions must have large norm. Currently, as far as we know, none of the conic optimization solvers indicate to the user that the feasibility of the problem is in question, they simply return an approximate solution or declare feasibility depending on their stopping criteria. Usually, if a problem is close to being infeasible that indicates that the model is not perfect, it is ill-posed. Either the data is wrong or the problem should be stated differently.

4.4 A library of infeasible problems

Problem libraries for linear, quadratic and general nonlinear optimization usually contain a good selection of *practical* infeasible problems. Unfortunately, the infeasible problems in the SDPLIB library (see [2]) or among he DIMACS Challenge problems [6] are rather small and trivial. A comprehensive library of infeasible problems would make testing and development of new stopping criteria easier.

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