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Advanced Optimization Laboratory

Title:

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AdvOl-Report No. 2007/11

July 2007, Hamilton, Ontario, Canada

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July 4, 2007

Abstract

Yu and Liu's strong duality theorem under the time-sharing property requires the Slater condition to hold for the considered general nonconvex problem, what is satisfied for the specific application. We further extend the scope of the theorem under Ky Fan convexity which is slightly weaker than Yu&Lui's time-sharing property.
Keywords: Strong duality theorem, time-sharing property, Ky Fan convexity.

1 Introduction

Recently, the multiuser spectrum problem for digital subscriber lines (DSL) draws much attention in the communication research community. The problem can be formulated as a nonconvex optimization problem. In 2006, the authors of [7] presented an optimization model for the multiuser power control optimization problem, and derived a strong duality theorem under the so-called time-sharing property for a generalized form of the problem. Based on that strong duality result, they presented dual methods to find a global optimal solution.

Unfortunately, as it is demonstrated by a counterexample, their strong duality theorem cannot hold without a certain regularity condition. According to their theorem, convexity of an optimization problem is sufficient to assure zero duality gap, since convex optimization problems naturally satisfy the time-sharing property introduced

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in [7]. For convex problems constraint qualifications, such as the Slater regularity condition, are sufficient to ensure a zero duality gap. We show in this note that the Slater regularity condition allows to prove an even stronger result.

2 Preliminaries

First of all, we recall the problem statement and necessary definitions from [7]. The optimization problem formulation of the multiuser spectrum for DSL is as follows:

$$\begin{aligned} \max \quad & \sum_{k=1}^{K} \omega_k \sum_{n=1}^{N} \log\left(1 + \frac{S_k^n}{\sigma_k^n + \sum_{j\neq k} \alpha_{jk}^n S_j^n}\right) \\ \text{s.t.} \quad & \sum_{n=1}^{N} S_k^n \leq P_k, \quad k = 1, \cdots K \\ & S_k^n \geq 0, \quad k = 1, \cdots, K, \; n = 1, \cdots, N, \end{aligned} \tag{1}$$

where S_k^n denotes user k's signal power over subchannel n, σ_k^n, and α_{jk}^n represent user k's noise power and the interference from user j to user k at subchannel n, respectively. Further, ω_k represents the relative weight given to user k, and P_k is the maximum power available to user k. Finally, K and N refer to the number of users of the communication system and the number of frequency carriers, respectively.

In the process of developing their duality theorem, the authors of [7] first generalized the optimization problem (1) into the following more general, partially separable optimization problem:

$$\begin{aligned} \max \quad & \sum_{n=1}^{N} \hat{f}_n(x_n) \\ \text{s.t.} \quad & \sum_{n=1}^{N} \hat{h}_n(x_n) \leq P, \end{aligned} \tag{2}$$

where $x_n \in R^K$, $n = 1, 2, \cdots, N$, are independent variables, $\hat{f}_n(\cdot)$: $R^K \to R$ and $\hat{h}_n(\cdot)$: $R^K \to R^L$ are given functions, and $P = (P_1, \cdots, P_L)^T$. All functions $\hat{f}_n$ and $\hat{h}_n$ are not necessarily convex.

We rewrite problem (2) into the standard form [1, 2] by setting $f_n(x_n) = -\hat{f}_n(x_n)$ and $h_n(x_n) = \hat{h}_n(x_n) - P$ for $n = 1, \cdots, N$:

$$(P) \qquad \begin{aligned} \min \quad & \sum_{n=1}^{N} f_n(x_n) \\ \text{s.t.} \quad & \sum_{n=1}^{N} h_n(x_n) \leq 0. \end{aligned}$$

The Lagrangian dual problem of (P) is given as follows:

$$(D) \qquad \max \quad \theta(\lambda) = \inf_x \left\{ \sum_{n=1}^{N} f_n(x_n) + \lambda^T \sum_{n=1}^{N} h_n(x_n) \right\}$$
$$\text{s.t.} \qquad \lambda \geq 0,$$

where $\lambda \in R^L$ is the vector of Lagrange multipliers.

Now for $y \in R^L$, we define the *perturbation problem* as follows [1]:

$$(P_y) \qquad \min \quad \sum_{n=1}^{N} f_n(x_n)$$
$$\text{s.t.} \quad \sum_{n=1}^{N} h_n(x_n) \leq y,$$

and define the perturbation function $\nu(y)$ as the optimal value of the perturbation problem (P_y) at y. The domain S of $\nu(y)$ is defined as

$$S = \left\{ y \in R^L : \exists x \text{ s.t.} \sum_{n=1}^{N} h_n(x_n) \leq y \right\}.$$

We restate the definition of the *time-sharing property* as follows [7]:

Definition 2.1 *Primal problem (P) is said to satisfy the* time-sharing property *if for any $\bar{y}$, $\hat{y} \in S$ and for any $0 \leq \alpha \leq 1$, for $y = \alpha\bar{y} + (1-\alpha)\hat{y}$, there always exists a feasible solution $x = (x_1, x_2, \cdots, x_N)$ with each $x_i \in R^K$, such that $\sum_{n=1}^{N} h_n(x_n) \leq y$ and $\sum_{n=1}^{N} f_n(x_n) \leq \alpha \sum_{n=1}^{N} f_n(\bar{x}_n^*) + (1-\alpha)\sum_{n=1}^{N} f_n(\hat{x}_n^*)$, where $\bar{x}^*$ and $\hat{x}^*$ are the corresponding optimal solutions.*

As pointed out in [7], if problem (P) is convex, then it satisfies the time-sharing property.

The following theorem is one of the main results in [7]. It claims that the time-sharing property implies zero duality gap.

Theorem 2.1 *If primal problem (P) satisfies the time-sharing property, then it has zero duality gap, i.e., primal problem (P) and dual problem (D) have the same optimal value.*

By the following counterexample, we show that this theorem is invalid. This example is due to R.J. Duffin [6]:

$$(P_1) \qquad \min \quad e^{-x_2}$$
$$\text{s.t.} \quad \sqrt{x_1^2 + x_2^2} - x_1 \leq 0.$$

For this problem, $(x_1, 0)$, for all $x_1 \geq 0$, is an optimal solution with optimal objective function value 1. The Lagrange dual of problem (P_1) is given by

$$(D_1) \qquad \max \quad \theta(\lambda) = \inf_x L(x, \lambda)$$
$$\text{s.t.} \quad \lambda \geq 0,$$

where $L(x, \lambda) = e^{-x_2} + \lambda\left(\sqrt{x_1^2 + x_2^2} - x_1\right)$. The optimal value of the dual problem is zero as $\theta(\lambda) = 0$ for all $\lambda \geq 0$. Hence, in spite of the fact that the primal problem is convex, and thus it satisfies the time-sharing property, a positive duality gap exists.

3 Generalized Convexity

First of all, we give the definition of Ky Fan Convexity [3, 4].

Definition 3.1 *The primal problem (P) is Ky Fan convex if for any $\bar{x}$, $\hat{x}$ and for any $0 \leq \alpha \leq 1$, there exists an x, such that*

$$\sum_{n=1}^{N} h_n(x_n) \leq \alpha \sum_{n=1}^{N} h_n(\bar{x}_n) + (1 - \alpha) \sum_{n=1}^{N} h_n(\hat{x}_n)$$

and

$$\sum_{n=1}^{N} f_n(x_n) \leq \alpha \sum_{n=1}^{N} f_n(\bar{x}_n) + (1 - \alpha) \sum_{n=1}^{N} f_n(\hat{x}_n).$$

The following proposition follows directly from the definitions of the time-sharing property and Ky Fan convexity.

Proposition 3.1 *The time-sharing property implies the Ky Fan convexity. Furthermore, if perturbation problem (P_y) attains an optimal solution for all $y \in S$, then the time-sharing property is equivalent to Ky Fan convexity.*

By this proposition Ky Fan convexity is a milder condition than the time-sharing property. Let us assume that the feasible set of the perturbation problem (P_y) is compact and the objective function is lower semicontinuous. Note that these properties hold for problem (1). Then the attainment of an optimal solution to the perturbation problem (P_y) is guaranteed. In this case, primal problem (P) satisfies the time-sharing property if and only if (P) is Ky Fan convex.

Finally in this section, we show the relation between Ky Fan convexity and the convexity of the perturbation function.

Proposition 3.2 *If primal problem (P) is Ky Fan convex, then its perturbation function $\nu(y)$is convex. With the existence of optimal solutions to the perturbation problem (P_y) for all $y \in S$, the reverse statement holds, too.*

This proposition is of importance in developing the new strong duality theorem. Its proof is in the APPENDIX.

4 Corrected and extended strong duality theorem

It is well-known that under the Slater regularity condition a convex optimization problem has zero duality gap, which is referred to as the Strong Duality Theorem of convex problems (see e.g., Theorem 6.2.4 [1]). Similarly, we claim that if an optimization problem satisfies both the Ky Fan convexity and the Slater condition, then the primal problem and the dual problem have the same optimal value, thus the duality gap is zero at optimality. Since primal problem (P) contains only inequality constrains, the *Slater regularity condition* is reduced to the existence of an $\hat{x} = (x_1, x_2, \cdots, x_N)^T$ with each $x_i \in R^K$ such that $\sum_{n=1}^{N} h_n(x_n) < 0$.

Before formally stating the revised theorem, we need the following corollaries.

Corollary 4.1 *(Theorem 5.2.6 [5]) If problem (P) satisfies the Slater regularity condition, then* 0 *is an interior point of the domain S of the corresponding perturbation function.*

Corollary 4.2 *(Theorem 4.1.3 [2]) If a function is convex, then it is continuous on the interior of its domain.*

Corollary 4.3 *Assume that the optimal value of primal problem (P) is finite. Then the duality gap is zero, i.e. the optimal values of primal problem (P) and dual problem (D) are equal, if and only if the perturbation function $\nu(y)$ is lower semicontinuous at 0.*

By Proposition 3.2, Corollary 4.1, Corollary 4.2 and Corollary 4.3, we immediately have the following revised and extended strong duality theorem:

Theorem 4.1 *If primal problem (P) satisfies both the Ky Fan convexity and the Slater regularity condition, then it has zero duality gap, i.e. the optimal objective function values of the primal problem and the dual problem are equal. Moreover, if the dual optimal value is finite, a dual optimal solution exists, i.e., the dual optimal value is attained.*

5 Discussion

Although Theorem 2.1 (Theorem 1 in [7]) does not hold in general, and thus a duality gap may occur for the general problem (P) (problem (4) in [7]), the claimed duality result holds for the specific multiuser spectrum optimization for DSL (1). This is the case because optimization problem (1) satisfies the Slater condition, since it contains only affine constraints that define a polyhedral set of feasible solutions with nonempty interior. Consequently, the dual methods proposed in [7] are applicable to the problem (1).

6 Appendix

Proof of Proposition 3.2: The second statement is obvious, thus we need to prove only the first one.

For any $\bar{y}$ and $\hat{y} \in S$, and for any $\varepsilon > 0$, by the definition of the perturbation function $\nu(y)$, there exist $\bar{x}$ and $\hat{x}$, such that,

$$\sum_{n=1}^{N} h_n(\bar{x}_n) \le \bar{y}, \qquad \sum_{n=1}^{N} h_n(\hat{x}_n) \le \hat{y},$$
$$\sum_{n=1}^{N} f_n(\bar{x}_n) \le \nu(\bar{y}) + \varepsilon, \qquad \sum_{n=1}^{N} f_n(\hat{x}_n) \le \nu(\hat{y}) + \varepsilon.$$

Since primal problem (P) is Ky Fan convex, $\alpha \in [0,1]$, there exists x, such that,

$$\begin{aligned}\sum_{n=1}^{N} h_n(x_n) &\le \alpha \sum_{n=1}^{N} h_n(\bar{x}_n) + (1-\alpha) \sum_{n=1}^{N} h_n(\hat{x}_n) \\ &\le \alpha\bar{y} + (1-\alpha)\hat{y}\end{aligned}$$

and

$$\begin{aligned}\sum_{n=1}^{N} f_n(x_n) &\le \alpha \sum_{n=1}^{N} f_n(\bar{x}_n) + (1-\alpha) \sum_{n=1}^{N} f_n(\hat{x}_n) \\ &\le \alpha(\nu(\bar{y}) + \varepsilon) + (1-\alpha)(\nu(\hat{y}) + \varepsilon) \\ &= \alpha\nu(\bar{y}) + (1-\alpha)\nu(\hat{y}) + \varepsilon.\end{aligned}$$

Therefore, $\nu(\alpha\bar{y} + (1-\alpha)\hat{y}) \le \alpha\nu(\bar{y}) + (1-\alpha)\nu(\hat{y})$. The proof is complete. ■

Acknowledgment

The authors would like to thank Tim Davidson for his constructive comments and suggestion on a semi-final version of this paper.

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