McMaster University

Advanced Optimization Laboratory

Title:

Notes on duality in second order and p-order cone optimization

Author:

Erling D. Andersen, Cornelis Roos and Tamás Terlaky

AdvOl-Report No. 2000/8

May 2000, Hamilton, Ontario, Canada

Notes on duality in second order and p-order cone optimization

E. D. Andersen[*], C. Roos[†], and T. Terlaky[‡]

April 26, 2000

Abstract

Recently, the so-called *second order cone optimization problem* has received much attention, because the problem has many applications and the problem can at least in theory be solved efficiently by interior-point methods.

In this note we treat duality for second order cone optimization problems and in particular whether a nonzero duality gap can be introduced when casting a convex quadratically constrained optimization problem as a second order cone optimization problem. Furthermore, we also discuss the p-order cone optimization problem which is a natural generalization of the second order case. Specifically, we suggest a new self-concordant barrier for the p-order cone optimization problem.

1 Introduction

The second order cone optimization problem can be stated as

$$\begin{array}{llrcl} (SOCP) & \text{minimize} & f^T x & & \\ & \text{subject to} & \|A^i x - b^i\| & \leq & c_{i:}x - d_i, \quad i = 1, \ldots, k, \\ & & Hx & = & h. \end{array}$$

where $A^i \in \mathbf{R}^{(m_i-1)\times n}$ and $H \in \mathbf{R}^{l\times n}$ and all the other quantities have conforming dimensions. $c_{i:}$ denotes the ith row of $C \in \mathbf{R}^{k\times n}$. $\|\cdot\|$ denotes the Euclidean norm. Clearly, the problem $(SOCP)$ is a convex but non-smooth problem because the norm is not differentiable at zero.

We will not survey the numerous applications for this optimization model here, but rather refer the reader to [4]. However, it can be observed that linear, convex quadratic, and convex quadratically constrained optimization problems all can be stated as second order cone optimization problems. For example if $m_i = 1$ for all i's, then $(SOCP)$ reduces to an ordinary linear optimization problem.

The outline of the paper is as follows. We first discuss duality for second-order cone optimization problems and present two examples which demonstrate that nonzero duality gap can occur. Next we show that if a convex quadratically constrained optimization problem is formulated as a second order cone optimization problem then a positive duality gap cannot occur. Finally, we discuss the p-order generalization of the second order cone case.

[*] TU Delft, Mekelweg 4, 2628 CD Delft, The Netherlands, Email: e.d.andersen@mosek.com.

[†] TU Delft, Mekelweg 4, 2628 CD Delft, The Netherlands, Email: c.roos@its.tudelft.nl.

[‡] McMaster University, Department of Computing and Software, Hamilton, Ontario, Canada. L8S 4L7. Email: terlaky@cas.mcmaster.ca. Part of this research was done while the author was employed at TU Delft.

2 Duality

The dual problem corresponding to $(SOCP)$ is

$$\begin{array}{llrcl} (SOCD) & \text{maximize} & b^T z + d^T w + h^T v & & \\ & \text{subject to} & A^T z + C^T w + H^T v & = & f, \\ & & \|z^i\| & \leq & w_i, \quad i = 1, \ldots, k, \end{array}$$

where $z_i \in \mathbf{R}^{m_i - 1}$ and $w \in \mathbf{R}^k$ and we use the notation that

$$A := \begin{bmatrix} A^1 \\ \vdots \\ A^k \end{bmatrix} \text{ and } z := \begin{bmatrix} z^1 \\ \vdots \\ z^k \end{bmatrix}.$$

The following duality theorem is well-known.

Theorem 2.1 *Let ν_{SOCP} denote the optimal objective value of $(SOCP)$ where minimize is replaced by* inf. *Similarly, let ν_{SOCD} denote the optimal objective value of $(SOCD)$ where maximize is replaced by* sup. *Then the following holds:*

1. *Weak duality: If both $(SOCP)$ and $(SOCD)$ are feasible, then $\nu_{SOCP} \geq \nu_{SOCD}$.*
2. *Strong duality: If either $(SOCP)$ or $(SOCD)$ is Slater regular, then $\nu_{SOCP} = \nu_{SOCD}$. Moreover, If both the primal and dual problem are Slater regular, then both problems have an optimal solution and $\nu_{SOCP} = \nu_{SOCD}$.*

(We use the convention $\nu_{SOCP} = \infty$ if $(SOCP)$ is infeasible. Similarly, $\nu_{SOCD} = -\infty$ if $(SOCD)$ is infeasible.)

Proof: [2] □

Subsequently, we will present some examples that demonstrates that strong duality in general requires the Slater regularity condition, because otherwise there might be a positive duality gap between the primal and dual problem.

2.1 Infinite duality gap

The first example is

$$\begin{array}{lrcl} \text{minimize} & -x_2 & & \\ \text{subject to} & \sqrt{x_1^2 + x_2^2} - x_1 & \leq & 0, \end{array} \tag{1}$$

which has $\{(x_1, x_2) : \ x_1 \geq 0, \ x_2 = 0\}$ as the set of feasible solutions. From here it is obvious that the optimal objective value ν_{SOCP} is zero and the problem is not Slater regular. The dual problem corresponding to (1) is

$$\begin{array}{lrcl} \text{maximize} & 0 & & \\ \text{subject to} & z_1 + w_1 & = & 0, \\ & z_2 & = & -1, \\ & \sqrt{z_1^2 + z_2^2} & \leq & w_1. \end{array} \tag{2}$$

The constraints of (2) can be reduced to

$$\sqrt{w_1^2 + 1} \leq w_1$$

which implies (2) is infeasible thus $\nu_{SOCD} = -\infty$. Hence, in this case the duality gap is infinity.

2.2 Finite but nonzero duality gap

Next consider the example

$$\begin{array}{lrcl} \text{minimize} & x_2 & & \\ \text{subject to} & \sqrt{x_1^2 + (x_2 - 1)^2} & \leq & x_1, \\ & \sqrt{(-x_1 + x_2)^2} & \leq & x_1. \end{array} \tag{3}$$

The first constraint clearly implies that $x_2 = 1$ in any feasible solution. Given that fact it follows from the second constraint that

$$1 \leq 2x_1.$$

Hence, the feasible set consists of $\{(x_1, x_2) : \ x_1 \geq \frac{1}{2}, \ x_2 = 1\}$ and the optimal objective value is 1. The dual problem corresponding to (3) is

$$\begin{array}{lrcl} \text{maximize} & z_2 & & \\ \text{subject to} & z_1 + w_1 - z_3 + w_2 & = & 0, \\ & z_2 + z_3 & = & 1, \\ & \sqrt{z_1^2 + z_2^2} & \leq & w_1, \\ & \sqrt{z_3} & \leq & w_2. \end{array} \tag{4}$$

Since, it follows from the two last constraints that

$$w_1 \geq |z_1| \text{ and } w_2 \geq |z_3|,$$

then

$$w_1 + z_1 \geq 0 \text{ and } w_2 - z_3 \geq 0.$$

Using the first constraint this implies

$$w_1 = -z_1 \text{ and } w_2 = z_3.$$

Now using the second constraint we have that

$$z_2 = 1 - z_3 = 1 - w_2.$$

Therefore, (4) is equivalent to

$$\begin{array}{lrcl} \text{maximize} & 1 - w_2 & & \\ \text{subject to} & \sqrt{w_1^2 + (1 - w_2)^2} & \leq & w_1, \\ & \sqrt{w_2^2} & \leq & w_2 \end{array} \tag{5}$$

which has the feasible set $\{(w_1, w_2) : \ w_1 \geq 0, \ w_2 = 1\}$ and the optimal objective value is zero. Hence, we have constructed an example where both the primal problem (2) and dual problem (4) has an optimal solution, but nevertheless the duality gap is nonzero.

The reader can verify that if $(x_2 - 1)^2$ is replaced by $(x_2 - \alpha)^2$ in the problem definition (3), then for any $\alpha > 0$ there will be a positive duality gap of size α.

2.3 Non-attainment

The example

$$\begin{array}{lrcl} \text{minimize} & x_2 & & \\ \text{subject to} & \left\| \begin{bmatrix} 1 \\ x_1 - x_2 \end{bmatrix} \right\| & \leq & x_1 \end{array} \tag{6}$$

shows that the optimal objective value is not always attained. The reason is that the constraint implies

$$1 + x_2^2 \leq 2x_1x_2. \tag{7}$$

This shows that the optimal objective value is zero but cannot be attained.

2.4 From quadratically constrained optimization to second order cone optimization

According to [4] then one of the advantages of second order cone optimization is that convex quadratically constrained optimization can be cast as a second order cone optimization. However, an issue not addressed in [4] is whether recasting a quadratically constrained optimization problem as a second order cone optimization problem can introduce positive duality gap. As demonstrated in the previous section a positive dualtity gap might occur. This would be very bad because it has already been demonstrated in [8, 9, 10] that there exist a dual problem corresponding to any convex quadratically constrained optimization problem which has zero duality gap.

We will therefore address the issue whether a convex quadratically constrained optimization problem recast as a second order cone optimization problem has worse duality properties than the originally problem.

Any convex quadratically constrained optimization problem can be stated on the form

$$\begin{array}{lrcl} \text{minimize} & c^T x & & \\ \text{subject to} & \frac{1}{2}||(Q^i)^T x||^2 - a_{i:}x + b_i & \leq & 0, \quad i = 1, \ldots, m, \end{array} \tag{8}$$

where $Q^i \in \mathbf{R}^{n \times l_i}$ and $A \in \mathbf{R}^{m \times n}$. The remaining quantities are assumed to have conforming dimensions. The ordinary Lagrange dual corresponding to (8) is identical to

$$\begin{array}{lrcl} \text{maximize} & b^T y - \sum_{i=1}^m \frac{1}{2} y_i ||(Q^i)^T x||^2 & & \\ \text{subject to} & c + \sum_{i=1}^m y_i Q^i (Q^i)^T x - A^T y & = & 0, \\ & y \geq 0. & & \end{array} \tag{9}$$

Now using the definition

$$z^i := y_i (Q^i)^T x$$

we obtain the alternative dual problem

$$\begin{array}{lrcl} \text{maximize} & b^T y - \sum_{i=1}^m \frac{1}{2} \frac{||z^i||^2}{y_i} & & \\ \text{subject to} & c + \sum_{i=1}^m Q^i z^i - A^T y & = & 0, \\ & y \geq 0, & & \end{array} \tag{10}$$

which appears in [10]. The reader can easily verify that (9) and (10) are equivalent. Observe that the dual problem does not contain any primal variables and has linear constraints only.

Using the duality theory developed for ℓ_p programming presented in [8, 9, 10] we obtain the proposition:

Proposition 2.1 *Given that both (8) and (10) has a feasible solution, then the duality gap between those problems are zero and (8) attains its optimum value. Moreover, the duality gap between (8) and (9) is zero as well.*

Proof: Given (8) and (10) both has a feasible solution then it follows from [10] that the duality gap between (8) and (10) is zero and (8) attains its optimum value.

Hence, we have that (10) has a feasible solution $(\hat{y}, \hat{z}^i)$ having bounded objective value i.e.

$$b^T\hat{y} - \sum_{i=1}^{m} \frac{1}{2} \frac{||\hat{z}^i||^2}{\hat{y}_i} > -\infty. \tag{11}$$

Obviously if the system

$$\sum_{i=1}^{m} \hat{y}_i Q^i (Q^i)^T x = \sum_{i=1}^{m} Q^i \hat{z}^i \tag{12}$$

has a solution, then (9) also has a solution. Assume the contrary is the case i.e. (12) does not have solution. This implies there exists a u such that

$$u^T \sum_{i=1}^{m} \hat{y}_i Q^i (Q^i)^T = 0 \tag{13}$$

and

$$u^T \sum_{i=1}^{m} Q^i \hat{z}^i \neq 0. \tag{14}$$

Now multiply both sides of (13) from the left by u and

$$u^T \sum_{i=1}^{m} \hat{y}_i Q^i (Q^i)^T u = \sum_{i=1}^{m} \hat{y}_i \left\|(Q^i)^T u\right\|^2 = 0$$

is obtained. Therefore,

$$\hat{y}_i \left\|(Q^i)^T u\right\| = 0, \quad \forall i$$

holds and hence $(Q^i)^T u = 0$ if $\hat{y}_i > 0$. On the other hand if $\hat{y}_i = 0$, then (11) implies $\hat{z}^i = 0$. The combination of these two facts implies

$$u^T Q^i \hat{z}^i = 0, \quad \forall i$$

which is a contradiction to (14). Therefore, we conclude if (10) is feasible, then (9) is feasible.

Now let $\hat{x}$ be any solution to (12), then

$$\begin{aligned} 0 &\leq \sum_{i=1}^{m} \frac{\left\|\hat{y}_i (Q^i)^T \hat{x} - \hat{z}^i\right\|^2}{\hat{y}_i} \\ &= \sum_{i=1}^{m} \frac{\left\|\hat{y}_i (Q^i)^T \hat{x}\right\|^2 + \left\|\hat{z}^i\right\|^2 - 2\hat{y}_i (\hat{z}^i)^T (Q^i)^T \hat{x}}{\hat{y}_i} \\ &= \sum_{i=1}^{m} \left(\frac{\left\|\hat{z}^i\right\|^2}{\hat{y}_i} + \hat{y}_i \left\|(Q^i)^T \hat{x}\right\|^2 \right) - 2 \sum_{i=1}^{m} (\hat{y}_i Q^i (Q^i)^T \hat{x})^T \hat{x} \\ &= \sum_{i=1}^{m} \left(\frac{\left\|\hat{z}^i\right\|^2}{\hat{y}_i} - \hat{y}_i \left\|(Q^i)^T \hat{x}\right\|^2 \right) \end{aligned} \tag{15}$$

and it follows

$$\sum_{i=1}^{m} \hat{y}_i \left\|(Q^i)^T \hat{x}\right\|^2 \leq \sum_{i=1}^{m} \frac{\|\hat{z}^i\|^2}{\hat{y}_i}. \tag{16}$$

Clearly, $(\hat{x}, \hat{y})$ is a feasible solution to (9) and (16) shows that this solution has the same or a better objective value than the feasible solution $(\hat{y}, \hat{z})$ to the problem (10).

In summary we have proved for any feasible solution to (10) with a bounded objective value, then there exists a feasible solution to (9) having the same or a better objective value. This implies that the duality gap between (8) and (9) is zero. □

It has been proved in [3] that problem (8) and problem (10) satisfies the self-concordant condition and thus both problems can be solved efficiently. Alternatively problem (8) can be recast as a second order cone optimization problem and solved as such. One way to perform the reformulation of (8) is as follows. First introduce two additional variables t and u to obtain

$$\begin{array}{lccll} \text{minimize} & c^T x & & & \\ \text{subject to} & \|(Q^i)^T x\|^2 & \leq & u_i^2 - t_i^2, & i = 1, \ldots, m, \\ & u - t & = & Ax - b, & \\ & u + t & = & 2e. & \end{array} \tag{17}$$

e is the vector of all ones. Note that this formulation implicitly contains the constraints

$$u^2 - t^2 = (u_i + t_i)(u_i - t_i) \geq 0 \quad \text{and} \quad u_i + t_i \geq 0$$

which implies that $u_i \geq 0$ in all feasible solutions. Problem (17) is equivalent to

$$\begin{array}{lccll} \text{minimize} & c^T x & & & \\ \text{subject to} & \left\| \begin{array}{c} (Q^i)^T x \\ t_i \end{array} \right\|^2 & \leq & u_i^2, & i = 1, \ldots, m, \\ & u & = & e + \frac{(Ax-b)}{2}, & \\ & t & = & e - \frac{(Ax-b)}{2} & \end{array} \tag{18}$$

which after elimination of the variables u and t leads to the second order cone optimization problem

$$\begin{array}{lccll} \text{minimize} & c^T x & & & \\ \text{subject to} & \left\| \begin{array}{c} (Q^i)^T x \\ 1 - \frac{(a_{i:}x - b_i)}{2} \end{array} \right\| & \leq & 1 + \frac{(a_{i:}x - b_i)}{2}, & i = 1, \ldots, m. \end{array} \tag{19}$$

The dual problem corresponding to (19) is

$$\begin{array}{llcl} \text{maximize} & -\frac{1}{2}(b + 2e)^T \bar{z} + \frac{1}{2}(b - 2e)^T w & & \\ \text{subject to} & \sum_{i=1}^{k} (Q^i) z^i - \frac{A^T w - A^T \bar{z}}{2} & = & c, \\ & \left\| \begin{array}{c} z^i \\ \bar{z}_i \end{array} \right\| & \leq & w_i, \quad 1, \ldots, m. \end{array} \tag{20}$$

Now our question can be stated as if there is a positive duality gap between the primal problem (19) and the dual problem (20)? The answer is given in the subsequent proposition.

Proposition 2.2 *Given both (8) and (9) have a feasible solution, then both problem (19) and problem (20) has a feasible solution. Moreover, the duality gap is zero.*

Proof: It should be obvious that any feasible solution to (8) essentially defines a feasible solution to (19) as well having the same objective value for both problems.

Moreover, given the assumptions then (9) has a feasible solution $(\hat{x}, \hat{y})$. Next let

$$\begin{array}{rcll} z^i & = & -\hat{y}_i (Q^i)^T \hat{x}, & i = 1, \ldots, m, \\ w & = & \hat{y}_i \left(\frac{\left\| (Q^i)^T \hat{x} \right\|^2}{4} + 1 \right), & i = 1, \ldots, m, \\ \bar{z} & = & w - 2\hat{y}. & \end{array}$$

which we claim is a feasible solution of (20). Clearly $w \geq 0$ and if these values are substituted into (20) and the resulting problem is simplified, then we obtain the problem

$$\begin{array}{lll} \text{minimize} & b^T \hat{y} - \sum_{i=1}^m \frac{1}{2} \hat{y}_i \left\| (Q^i)^T \hat{x} \right\|^2 & \\ \text{subject to} & c + \sum_{i=1}^m \hat{y}^i (Q^i)(Q^i)^T x - A^T \hat{y} & = \; 0, \\ & \hat{y} & \geq \; 0. \end{array} \tag{21}$$

Obviously, (9) and (21) are identical. Hence, we have demonstrated that any feasible solution to (9) can easily be converted to a feasible solution to (21). Moreover, this leaves the objective value of the solution unchanged.

In conclusion given the duality gap between (8) and (9) is zero, then the duality gap between the problems (19) and (20) is zero as well. Hence, nothing is lost (duality wise) by reformulating a quadratically constrained optimization problem as a second order cone optimization problem. □

Finally, observe that the dual problem (10) is equivalent to the following second order cone optimization problem

$$\begin{array}{lll} \text{maximize} & b^T y - \sum_{i=1}^m t_i & \\ \text{subject to} & c + \sum_{i=1}^m Q^i z^i - A^T y & = \; 0, \\ & ||z^i||^2 & \leq \; 2 y_i t_i, \\ & y \geq 0 & \end{array} \tag{22}$$

involving the so-called rotated quadratic cone.

One obvious question is which of the many formulations of the convex quadratically constrained optimization can be solved most efficiently. The answer is that an ϵ-optimal solution can be obtained in $O(\sqrt{m}\frac{1}{\epsilon})$ Newton steps for the problems (8), (19), and (10) using an interior-point algorithm. This is proved in [5], [7], and [3]

Therefore, from a complexity point of view all three formulations of the quadratically constrained optimization problem is equally difficult to solve. However, only the second order cone optimization formulation has the special property that the problem is self-dual (see next section) and the cone is homogeneous [7]. This implies that efficient primal-dual algorithms exist for this class of problems and not for the other formulations [7]. Hence, casting a convex quadratically optimization problem as second order cone optimization problem and solving it using a primal-dual algorithm might be the most efficient way. However, this still has to be verified in practice.

3 p-order cone optimization

A generalization of the second order optimization model is the p-order cone optimization problem which can be expressed as follows

$$\begin{array}{llrcl} \text{(POCP)} & \text{minimize} & c^T x & & \\ & \text{subject to} & Ax & = & b, \\ & & x^i & \in & K_i, \quad i = 1, \ldots, k, \end{array}$$

where $A \in \mathbf{R}^{m \times n}$. Moreover, let $x^i \in \mathbf{R}^{n_i}$ and

$$x := \begin{bmatrix} x^1 \\ x^2 \\ \vdots \\ x^k \end{bmatrix}.$$

In this case we use the definition

$$K_i := \left\{ x^i \in \mathbf{R}^{n_i} : \; x^i_1 \geq \left(\sum_{j=2}^{n_i} |x^i_j|^{p_i} \right)^{\frac{1}{p_i}} \right\}. \tag{23}$$

Given $p_i > 1$ and

$$\frac{1}{p_i} + \frac{1}{q_i} = 1$$

then the dual cone corresponding to K_i is

$$K_i^* := \left\{ s^i \in \mathbf{R}^{n_i} : \; s^i_1 \geq \left(\sum_{j=2}^{n_i} |s^i_j|^{q_i} \right)^{\frac{1}{q_i}} \right\}.$$

The second order case is when $p = 2$ and in that case the primal and dual cone is identical i.e. self-dual.

The dual problem corresponding to $(POCP)$ is

$$\begin{array}{llrcl} \text{(DOCP)} & \text{maximize} & b^T y & & \\ & \text{subject to} & A^T y + s & = & c, \\ & & s^i & \in & K_i^*, \quad i = 1, \ldots, k, \end{array}$$

where s^i and s is constructed as x^i and x. This has been shown in [1].

Solution of the p-order cone problem has already been studied by Xue and Ye [11]. Indeed they develop several polynomial time algorithm using different self-concordant barriers for the cone.

Subsequently, we present a new self-concordant barrier for the p-order cone which has a better barrier parameter than the one suggested by Xue and Ye [11]. Moreover, we will demonstrate that both $(POCP)$ and $(DOCP)$ can be reformulated as ordinary smooth convex programs. This has the advantage that the problems can be solved using existing optimization software. Although it might not be as efficient as using special purpose algorithms.

By definition the constraint $(r, x) \in K$ where K has the form (23) is equivalent to

$$\|x\|_p \leq r \tag{24}$$

where we assume that $x \in \mathbf{R}^l$ and $r \in \mathbf{R}$. Although this constraint expresses a convex set then it is nonsmooth, because the norm is not differentiable at zero. However, it is easy to verify that constraint (24) can be replaced by the constraints

$$0 \leq r, \ \ \|x\|^p - r^p \leq 0, \tag{25}$$

which in turn is equivalent to the constraints

$$0 \leq r, \ \frac{\|x\|^p}{r^{p-1}} - r \leq 0. \tag{26}$$

Note in particular for the second order case i.e. $p = 2$, then the function

$$\frac{\|x\|^2}{r}$$

is a smooth convex function on its domain $\{(r, x) : r > 0\}$. Moreover, it can be proven that

$$-\ln(r - r^{-1}\|x\|^2) - \ln(r) = \ln(r^2 - \|x\|^2)$$

is a 2-self-concordant barrier for the constraint (26). This implies in the second order case, that the cone constraints can be replaced by "ordinary" convex constraints and the resulting program can be solved in polynomial time using a standard interior-point algorithm.

On the other hand if p is odd and larger than 2, then the constraint (26) is not a smooth convex constraint. However, the constraints (26) be replaced by

$$\begin{array}{rcll} \sum_{i=1}^{l} t_i & \leq & r, & \\ |x_i|^p r^{1-p} & \leq & t_i, & i = 1, \dots, l, \\ r, t_i & \geq & 0, & i = 1, \dots, l, \end{array}$$

which is identical to

$$\begin{array}{rcll} \sum_{i=1}^{l} t_i & \leq & r, & \\ |x_i| & \leq & t_i^{\frac{1}{p}} r^{1-\frac{1}{p}}, & i = 1, \dots, l, \\ r, t_i & \geq & 0, & i = 1, \dots, l. \end{array}$$

Using the usual trick by introducing some additional constraints, then we can get rid of the absolute sign as follows

$$\begin{array}{rcll} \sum_{i=1}^{l} r_i & \leq & t, & \\ x_i - t_i^{\frac{1}{p}} r^{1-\frac{1}{p}} & \leq & 0, & i = 1, \dots, l, \\ -x_i - t_i^{\frac{1}{p}} r^{1-\frac{1}{p}} & \leq & 0, & i = 1, \dots, l, \\ r, t_i & \geq & 0, & i = 1, \dots, l. \end{array} \tag{27}$$

Clearly, l new constraints and variables has been introduced into the problem. Moreover, we have the following proposition

Proposition 3.1 *The set*

$$\left\{ (x_i, r_i, t) : \begin{array}{rcl} x_i - t_i^{\frac{1}{p}} r^{1-\frac{1}{p}} & \leq & 0, \\ -x_i - t_i^{\frac{1}{p}} r^{1-\frac{1}{p}} & \leq & 0, \\ t_i, r & \geq & 0 \end{array} \right\}$$

is a convex set and the function

$$-\ln(t_i^{\frac{1}{p}} r^{1-\frac{1}{p}} - x_i) - \ln(t_i^{\frac{1}{p}} r^{1-\frac{1}{p}} + x_i) - \ln(t) - \ln(r_i)$$

is a 4*-self-concordant barrier for this set.*

Proof: See Lemma 6 in [3]. □

We are now ready to state the main theorem

Theorem 3.1 *The function*

$$-\ln(r - \sum_{i=1}^{l} t_i) - \sum_{i=1}^{l} (-\ln(t_i^{\frac{1}{p}} r^{1-\frac{1}{p}} - x_i) - \ln(t_i^{\frac{1}{p}} r^{1-\frac{1}{p}} + x_i) - \ln(t) - \ln(r_i))$$

is a $(1 + 4l)$*-self-concordant barrier for the set given by* (27).

Proof: It follows directly from Proposition 3.1 and the barrier calculus outlined in [6, p. 292]. □

The best barrier presented in Xue and Ye [11] for the p-order cone where p is arbirary large has the parameter $200l$. Xue and Ye also presents another barrier, but it depends on p. Hence, the new barrier function is better. Although independent of the barrier then a short-step interior-point algorithm will solve the problem in $O(\sqrt{n} \ln(1/\epsilon))$ Newton steps.

4 Conclusion

In this paper we have shown that when a convex quadratically constrained optimization problem is cast as a second order cone optimization problem using the method outlined in Section 2.4 then the resulting primal and dual second order cone optimization problem has zero duality gap.

Finally, we discuss the p-order cone optimization problem and suggest a new self-concordant barrier function for the problem which has a better parameter than the one suggested in [11].

References

[1] I. Adler and F. Alizadeh. Primal-dual interior point algorithms for convex quadratically constrained and semidefinite optimization problems. Technical Report RRR-111-95, RUTCOR, Rutgers Center for Operations Research, P.O. Box 5062, New Brunswick, New Jersey, 1995.

[2] A. Ben-Tal and A Nemirovski. Convex optimization in engineering: Modeling, analysis, algorithms. 1999.

[3] D. den Hertog, F. Jarre, C. Roos, and T. Terlaky. A sufficient conditions for self-concordance, with application to some classes of structured convex programming problems. *Math. Programming*, 69(1):75–88, 1995.

[4] M.S. Lobo, L. Vandenberghe, S. Boyd, and H. Lebret. Second-order cone programming. Technical report, ISL, Electrical Engineering Department, Stanford University, Stanford, CA, 1997. Submitted to Linear Algebra and Applications.

[5] Y. Nesterov and A. Nemirovskii. *Interior-Point Polynomial Algorithms in Convex Programming*. SIAM, Philadelphia, PA, 1 edition, 1994.

[6] Yu. Nesterov. Interior-point methods: An old and new approach to nonlinear programming. *Math. Programming*, 79:285–297, 1998.

[7] Yu. Nesterov and J.-Ph. Vial. Homogeneous analytic center cutting plane methods for convex problems and variational inequalities. Technical Report 1997.4, Logilab, HEC Geneva, Section of Management Studies, University of Geneva, jul 1997.

[8] E. L. Peterson and J. G. Ecker. Geometric programming: Duality in quadratic programming and ℓ_p approximation ii. *J. on Appl. Math.*, 13:317–340, 1967.

[9] E. L. Peterson and J. G. Ecker. Geometric programming: Duality in quadratic programming and ℓ_p approximation iii. *J. Math. Anal. Appl.*, 29:365–383, 1970.

[10] T. Terlaky. On ℓ_p programming. *European J. Oper. Res.*, 22:70–100, 1985.

[11] G. Xue and Y. Ye. An Efficient Algorithm for Minimizing a Sum of P-Norms. Technical report, Department of Computer Science and Electrical Engineering, The University of Vermont, September 1997.

[12] G. Xue and Y. Ye. An efficient algorithm for minimizing a sum of euclidean norms with apllications. *SIAM J. on Optim.*, 7(4):1017–1036, 1997.